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# ON CONTRACTIONS OF HYPERGEOMETRIC FUNCTIONS ASSOCIATED WITH ROOT SYSTEMS

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Abstract. In this paper we extend previous classes of generalized Bessel functions. This follows via a limit transition from Heckman-Opdam's results in the theory of hypergeometric functions. Our starting point is to obtain the Bessel functions associated with Cartan motion groups by means of Harish-Chandra's spherical functions. Bessel functions related to noncompact causal symmetric spaces are also investigated. This paper is a survey of recent results in [B-Ø1] and [B-Ø2].

## 1. Introduction

A basic problem in harmonic analysis and applications is to study the Fourier transforms of invariant measures on submanifolds of Euclidean space. For example, let $d\mu$ be the $SO(n)$-invariant probability measure on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and consider

$$\psi(y) = \int_{\mathbb{S}^{n-1}} e^{i\langle x,y\rangle}\, d\mu(x), \qquad (\mathcal{E}_1)$$

with $\langle \cdot, \cdot \rangle$ the usual inner product. It is well-known that $(\mathcal{E}_1)$ can be found explicitly in terms of the modified Bessel functions of the first kind

$$I_\nu(r) = \sum_{k=0}^{\infty} \frac{(r/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}. \qquad (\mathcal{E}_2)$$

Indeed, $\psi(y) = \Gamma(n/2)\,(i|y|/2)^{-\frac{n}{2}+1}\, I_{\frac{n}{2}-1}(i|y|)$, and for $n$ odd, these functions are elementary (they may be expressed in terms of polynomials, exponential functions, or their derivatives).

In this paper we will study integrals of type $(\mathcal{E}_1)$, and special functions as in $(\mathcal{E}_2)$ needed to express them. The main tools are the Hypergeometric functions associated with root systems, and a deformation principle which we now explain in a simple case. It is well-known that the polynomial solutions of Laplace's differential equation on $\mathbb{R}^3$ which are symmetric around the $z$-axis and analytic in a neighborhood of the origin, can be expressed in spherical coordinates $(r, \theta, \phi)$ in the form $r^n P_n(\cos\theta)$. Here $P_n$ is the Legendre polynomial of order $n = 0, 1, 2, \ldots$. Now, consider the nature of the structure of spheres, cones, and planes associated with spherical coordinates in a region of space far from the origin and near the $z$-axis. The spheres approximate to planes and the cones approximate to cylinders, and the structure resembles the one associated with cylindrical coordinates $(\rho, \phi, z)$. The solutions of Laplace's equation referred to such coordinates are of the form $e^{\pm kz} J_0(k\rho)$, where $J_0$ is the Bessel function of the first kind of order 0, and $k$ is any constant. Recall that $J_\nu(r) = i^\nu I_\nu(-ir)$. It is therefore to be expected that, when $r$ and $n$ are large and $\theta$ is small in such a way that $r \sin\theta (= \rho)$ remains bounded, the Legendre polynomial should approximate to a Bessel function. This is equivalent to expect Bessel functions as limits of Legendre polynomials. In 1868, this type of limit was proved by Mehler as follows

$$\lim_{n\to\infty} P_n\Big(\cos\frac{\theta}{n}\Big) = J_0(\theta). \qquad (\mathcal{E}_3)$$

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The same formula also appeared in a work of Heine, done independently at about the same time, and it is nowadays known as the Mehler-Heine formula. This result has been extended to generalized Legendre polynomials by Heine and Rayleigh. For more recent treatments, see [Cl] and [Sta].

In the original proof of $(\mathcal{E}_3)$, the parameter $n$ is assumed to tend to infinity through integral values. One can also prove it when $n$ goes to infinity as a continuous real parameter.

Now, let us give another interpretation of the limit formula $(\mathcal{E}_3)$ in terms of spherical functions associated to noncompact symmetric spaces. In analogy to $(\mathcal{E}_3)$, we get

$$\lim_{\varepsilon\to 0} P_{\frac{i\lambda}{\varepsilon}}(\operatorname{ch}\varepsilon z) = J_0(\lambda z), \quad \lambda \in \mathbb{R}.$$

The functions $z \mapsto P_{i\lambda}(\operatorname{ch} z)$ are the spherical functions of the noncompact Riemannian symmetric space $SO_0(2,1)/SO(2)$. The functions $x \mapsto J_0(\lambda|x|)$ are the Bessel functions on the tangent space at the origin of $SO_0(2,1)/SO(2)$, viewed as a flat symmetric space. Thus, we may expect Bessel functions on flat symmetric spaces to be expressed as limits of spherical functions on the corresponding Riemannian symmetric spaces. This was the starting point of our investigation on Bessel functions on flat symmetric spaces by means of Harish-Chandra's spherical functions. In [B-Ø1] (see Section 2 below), we prove the following statement which can be seen as a generalization of the Mehler-Heine formula:

*The Bessel functions on flat symmetric spaces can be obtained as limits of Harish-Chandra's spherical functions on Riemannian symmetric spaces of noncompact type.* $(\mathcal{S})$

In this setting, the integral representation of the Bessel functions is sometimes called Harish Chandra–Itzykson–Zuber (HIZ)-type integral.

We mention that the contraction principle goes back to E. Inonu and E.P. Wigner [E-W], where the authors investigate, in several cases, relationships between representations of a Lie algebra and those of its appropriate limit Lie algebra. Since then, this principle has been carried out by other authors in different settings. See for instance [Cl], [D-R], [Ok-Ol2], and [Sta].

The advantage of the above described approach is that we can derive at least the same amount of explicit information for the Bessel functions, by a limit analysis, as for Harish-Chandra's spherical functions; in some cases even more information is attained. An important motivation to study Bessel functions originates in their relevance for the analysis of quantum many body systems of Calogero-Moser type, and in connection with the study of random matrices. See for example [Br-Hi], [Ols-Pe], and [Ol-Ve].

After this generalization of the Mehler-Heine type formula in the case of noncompact Riemannian symmetric spaces, we move to other directions were we may expect a statement similar to $(\mathcal{S})$ to hold.

In recent years, Harish-Chandra's theory of spherical functions on Riemannian symmetric spaces has been generalized in three different directions. In the 80s Heckman and Opdam extended the theory of Harish-Chandra to multi-variables hypergeometric functions associated with root systems and depending on additional parameters, namely the multiplicities (cf. [He-O], [O1], [O2]). Their construction was motivated by the theory of special functions. In one variable, Harish-Chandra's spherical functions on Riemannian symmetric spaces of rank-one are special instances of the Gaussian hypergeometric functions. This was the starting point of Heckman-Opdam's theory of hypergeometric functions. Harish-Chandra's spherical functions can always be regained by specializing the multiplicities. Another line of study consists of extending the theory of Harish-Chandra to a class of non-Riemannian symmetric spaces, called noncompact causal symmetric spaces. This was done in 1994 by Faraut, Hilgert, and Ólafsson [F-H-Ó]. Recently, Pasquale has presented an extension of Heckman-Opdam theory, which also includes the theory of Faraut-Hilgert-Ólafsson, by introducing the so-called $\Theta$-hypergeometric functions [P].

Using these developments in the theory of spherical functions, in [B-Ø2] (see Section 3, 4, and 5 below), we were able to extend the statement $(\mathcal{S})$ to the above mentioned three directions, namely to Bessel functions related to root systems, to Bessel functions associated with noncompact causal symmetric spaces, and finally to what we shall call the Θ-Bessel functions. We should note here that in [O3] Opdam also investigated the Bessel functions related to root systems from another point of view.

The main results of the present paper are: Theorems 2.3, 2.7, 3.3, 3.7, 4.3, and 5.4.

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## 2. Bessel functions on flat symmetric spaces

In this section we shall initiate an investigation of the so-called HIZ-type integral (Harish Chandra-Itzykson-Zuber); these are Fourier transforms of orbits of $K$ in the tangent space at the origin of a semi-simple noncompact symmetric space $G/K$, where $G$ is a connected noncompact semi-simple Lie group with finite center, and $K$ is a maximal compact subgroup. These integrals play a role in the theory of integrable systems in physics, and in connection with the study of random matrices. It is well-known that they correspond to spherical functions on the tangent space, viewed as a flat symmetric space. Our point of view is to see the HIZ-type integrals as limits of spherical functions for $G/K$, and we are able to obtain new and explicit formulas by analyzing the deformation (as the curvature goes to zero) of $G/K$ to its tangent space. We refer to [B-Ø1] for more details and information.

Let $G$ be a connected semisimple Lie group with finite center, and let $K$ be a maximal compact subgroup of $G$. The quotient manifold $G/K$ can be endowed with the structure of Riemannian symmetric space of the noncompact type.

Let $\theta : G \to G$ be the Cartan involution on $G$ corresponding to $K$, i.e. $K = \big\{k \in G \mid \theta(k) = k\big\}$. Denote by the same letter the derived involution $\theta : \mathfrak{g} \to \mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\theta$, where $\mathfrak{k} = \big\{X \in \mathfrak{g} \mid \theta(X) = X\big\}$ and $\mathfrak{p} = \big\{X \in \mathfrak{g} \mid \theta(X) = -X\big\}$.

Assume that $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{p}$, and let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$. Fix a Weyl positive chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and let $\Sigma^+$ be the corresponding set of positive roots in $\Sigma$. For $\alpha \in \Sigma$, let $\mathfrak{g}^{(\alpha)} := \big\{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\big\}$ be the associated root space, and set $m_\alpha := \dim(\mathfrak{g}^{(\alpha)})$. We denote by $\rho := (1/2)\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$. Let $\mathfrak{n}$ be the sum of the root spaces corresponding to the positive roots. The connected subgroups of $G$ associated with the subalgebras $\mathfrak{a}$ and $\mathfrak{n}$ are denoted by the corresponding capital letters. We have the Iwasawa decomposition $G = KAN$, and the Cartan decomposition $G = KAK$. For $g \in G$, define $H(g) \in \mathfrak{a}$ by $g \in K \exp(H(g))N$.

Denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathfrak{a}$ induced by the Killing form $B(\cdot, \cdot)$ of $\mathfrak{g}$, and let $\Pi$ be the fundamental system of simple roots associated with $\Sigma^+$. Denote by $\mathscr{W}_\Pi$ the Weyl group generated by the reflections $r_\alpha : \mathfrak{a}^* \to \mathfrak{a}^*$, with $\alpha \in \Pi$ and $r_\alpha(\lambda) := \lambda - 2\langle\lambda, \alpha\rangle\alpha/\langle\alpha, \alpha\rangle$. The action of $\mathscr{W}_\Pi$ on $\mathfrak{a}^*$ extends to $\mathfrak{a}$ and to the complexifications $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}^*_\mathbb{C}$.

Let $\mathscr{D}(G/K)$ be the algebra of $G$-invariant differential operators on $G/K$. Suppose the smooth complex-valued function $\varphi_\lambda$ is an eigenfunction of each $D \in \mathscr{D}(G/K)$,

$$D\varphi_\lambda = \gamma_D(\lambda)\varphi_\lambda, \qquad \lambda \in \mathfrak{a}^*_\mathbb{C}. \tag{2.1}$$

Here the eigenfunction is labeled by the parameters $\lambda$, and $\gamma_D(\lambda)$ is the eigenvalue. If in addition $\varphi_\lambda$ satisfies $\varphi_\lambda(\mathbf{e}) = 1$, where $\mathbf{e}$ is the identity element, and $\varphi_\lambda(kgk') = \varphi_\lambda(g)$ for $k, k' \in K$, then the function $\varphi_\lambda$ is called a spherical function. In [HC1], Harish-Chandra proves the following integral representation of the spherical functions.

**Theorem 2.1.** (cf. [HC1]) *As $\lambda$ runs through $\mathfrak{a}_{\mathbb{C}}^*$, the functions*

$$\varphi_\lambda(g) = \int_K e^{(i\lambda-\rho)H(gk)}\, dk, \qquad g \in G,$$

*exhaust the class of spherical functions on $G$. They are real analytic functions of $g \in G$ and holomorphic functions of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Moreover, two such functions $\varphi_\lambda$ and $\varphi_\mu$ are identical if and only if $\lambda = \omega\mu$, for some $\omega$ in the Weyl group $\mathscr{W}_\Pi$.*

Since $\mathfrak{a}^+$ is the interior of a fundamental domain of $\mathscr{W}_\Pi$, the Cartan decomposition implies that $\varphi_\lambda$ is uniquely determined by its restriction to $A^+ := \exp(\mathfrak{a}^+)$. Moreover, a $K$-bi-invariant function $\varphi_\lambda$ is an eigenfunction for $\mathscr{D}(G/K)$ if and only if its restriction to $A^+$ is an eigenfunction for the system of equations on $A^+$ given by the radial components of operators from $\mathscr{D}(G/K)$.

Let $\{H_i\}_{i=1}^N$ be a fixed orthonormal basis of $\mathfrak{a}$. For $H \in \mathfrak{a}$, denote by $\partial(H)$ the corresponding directional derivative in $\mathfrak{a}$. Let $\Delta(m)$ be the radial part of the Laplace-Beltrami operator on $G/K$. Then

$$\Delta(m) = \sum_{i=1}^N \partial(H_i)^2 + \sum_{\alpha\in\Sigma^+} m_\alpha \coth(\alpha)\partial(A_\alpha),$$

where $A_\alpha \in \mathfrak{a}$ is determined by $B(A_\alpha, H) = \alpha(H)$ for $H \in \mathfrak{a}$. In particular

$$\Delta(m)\varphi_\lambda = -\big(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle\big)\varphi_\lambda. \tag{2.2}$$

For $\varepsilon > 0$, denote by $\mathfrak{g}_\varepsilon$ the Lie algebra $\mathfrak{k} \oplus \mathfrak{p}$ with the Lie bracket $[\cdot,\cdot]_\varepsilon$ such that

$$\begin{aligned} &[X, X']_\varepsilon := [X, X'] && (X, X' \in \mathfrak{k}),\\ &[Y, Y']_\varepsilon := \varepsilon^2[Y, Y'] && (Y, Y' \in \mathfrak{p}),\\ &[X, Y]_\varepsilon := [X, Y] && (X \in \mathfrak{k},\ Y \in \mathfrak{p}). \end{aligned}$$

Here $[\cdot,\cdot]$ denotes the Lie bracket associated with $\mathfrak{g}$. The following map $\Phi_\varepsilon : \mathfrak{g}_\varepsilon \to \mathfrak{g}$ defined by

$$\Phi_\varepsilon(X) := X \ \text{ if } X \in \mathfrak{k}, \qquad \text{and} \qquad \Phi_\varepsilon(Y) := \varepsilon^{-1}Y \ \text{ if } Y \in \mathfrak{p},$$

is an isomorphism from $\mathfrak{g}_\varepsilon$ to $\mathfrak{g}$. Further, if $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ then $\varepsilon\alpha \in \Sigma(\mathfrak{g}_\varepsilon, \mathfrak{a})$, and the corresponding root space $\mathfrak{g}_\varepsilon^{(\varepsilon\alpha)}$ is given by

$$\mathfrak{g}_\varepsilon^{(\varepsilon\alpha)} = \Big\{ \varepsilon X_k + X_p \ \mid\ X_k + X_p \in \mathfrak{g}^{(\alpha)} \text{ where } X_k \in \mathfrak{k},\ X_p \in \mathfrak{p} \Big\}.$$

Let $G_\varepsilon$ be the analytic Lie group with Lie algebra $\mathfrak{g}_\varepsilon$ via the Baker-Campbell-Hausdorff formula. Denote by $\Delta^{(\varepsilon)}(m)$ the radial part of the Laplace-Beltrami operator on $G_\varepsilon/K$ given by

$$\Delta^{(\varepsilon)}(m) = \sum_{i=1}^N \partial(H_{i,\varepsilon})^2 + \sum_{\alpha\in\Sigma^+} m_\alpha \coth(\varepsilon\alpha)\partial(A_{\varepsilon\alpha}),$$

where $\{H_{i,\varepsilon}\}_{i=1}^N$ is a fixed orthonormal basis of $\mathfrak{a}$ in $\mathfrak{g}_\varepsilon$ (chosen as in (2.3) below), and $A_{\varepsilon\alpha} \in \mathfrak{a}$ is determined by $B_\varepsilon(A_{\varepsilon\alpha}, H) = \varepsilon\alpha(H)$ for $H \in \mathfrak{a}$. Here $B_\varepsilon(\cdot,\cdot)$ is the Killing form of $\mathfrak{g}_\varepsilon$. The above relationship between $\mathfrak{g}^{(\alpha)}$ and $\mathfrak{g}_\varepsilon^{(\varepsilon\alpha)}$ yields the fact that $B_\varepsilon(A_{\varepsilon\alpha}, H) = \varepsilon^2 B(A_{\varepsilon\alpha}, H)$, which implies

$$A_{\varepsilon\alpha} = \varepsilon^{-1}A_\alpha, \quad \text{and} \quad H_{i,\varepsilon} = \varepsilon^{-1}H_i. \tag{2.3}$$

The following theorem holds.

**Theorem 2.2.** (cf. [B-Ø1]) *Let*

$$\Delta^\circ(m) := \sum_{i=1}^N \partial(H_i)^2 + \sum_{\alpha\in\Sigma^+} \frac{m_\alpha}{\alpha}\partial(A_\alpha).$$

*Under the weak topology the following limit holds*

$$\lim_{\varepsilon\to 0}\varepsilon^2\Delta^{(\varepsilon)}(m)=\Delta^\circ(m).$$

By (2.3), one may check the relation $\Delta^{(\varepsilon)}(m)=(\varepsilon^*)^{-1}\circ\Delta(m)\circ(\varepsilon^*)$, where $\varepsilon^* f(X):=f(\varepsilon X)$, whilst $\varphi_\lambda$ is an eigenfunction for $\Delta(m)$. This observation suggests to obtain the eigenfunctions of $\Delta^\circ(m)$ as an appropriate limit of Harish-Chandra's spherical functions, which turns out to be true.

In terms of symmetric spaces, consider the limit $\varepsilon\to 0$ is equivalent to letting the curvature of $G/K$ tend to zero. This scaling removes the curvature so that in the limit we recover the tangent space at the origin of $G/K$, viewed as a flat symmetric space in the following sense. The symmetric spaces fall into three different categories: the compact-type, the noncompact-type, and Euclidean-type (which corresponds to flat symmetric spaces). The three cases can be distinguished by means of their curvature. In the class of compact-type, the symmetric space has sectional curvature everywhere positive. In the class of noncompact-type, the symmetric space has sectional curvature everywhere negative, and in the class of flat symmetric spaces, the sectional curvature is zero. Actually, if $\mathfrak{g}$ is the Lie algebra of a Lie group $G$ with the Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, then with $G_0:=K\ltimes\mathfrak{p}$, the flat symmetric space $G_0/K$ can be identified with $\mathfrak{p}$. The elements $g_0=(k,p)\in G_0$ act on $G_0/K$ in the following way:

$$g_0(p')=\mathrm{Ad}(k)p'+p,\qquad k\in K,\quad p,p'\in G_0/K.$$

Next we will see that the spherical functions associated with flat symmetric spaces can be obtained from the spherical functions on symmetric spaces of the noncompact-type, or the compact-type, by letting the curvature tend to zero from the left, or from the right, respectively. This fact is well-known.

For $\varepsilon>0$, write $g_\varepsilon=k\exp(\varepsilon X)$ where $k\in K$ and $X\in\mathfrak{p}$, and define the map $\varepsilon^*$ by $\varepsilon^* f(k\exp(X)):=f(k\exp(\varepsilon X))$. Using the fact that $\Delta^{(\varepsilon)}(m)=(\varepsilon^*)^{-1}\circ\Delta(m)\circ(\varepsilon^*)$, and that $\varphi_\lambda$ is an eigenfunction of $\Delta(m)$ with the eigenvalue $-(\langle\lambda,\lambda\rangle+\langle\rho,\rho\rangle)$, we obtain

$$\Delta^{(\varepsilon)}(m)\varphi_{\frac{\lambda}{\varepsilon}}(g_\varepsilon)=-\Big(\Big\langle\frac{\lambda}{\varepsilon},\frac{\lambda}{\varepsilon}\Big\rangle+\Big\langle\rho,\rho\Big\rangle\Big)\varphi_{\frac{\lambda}{\varepsilon}}(g_\varepsilon).$$

Denote by

$$\psi(\lambda,X):=\lim_{\varepsilon\to 0}\varphi_{\frac{\lambda}{\varepsilon}}(g_\varepsilon).$$

Now, we summarize the consequence of all the above discussions in the light of Theorem 2.1 and Theorem 2.2.

**Theorem 2.3.** (cf. [D-R], [B-Ø1]) *The limit $\psi(\lambda,X)$, and its derivatives exist. Its integral representation is given by*

$$\psi(\lambda,X)=\int_K e^{iB(\mathrm{Ad}(k)X,A_\lambda)}\,dk,\qquad \textit{for } (\lambda,X)\in\mathfrak{a}^*_{\mathbb{C}}\times\mathfrak{p}.$$

*Moreover, $\psi(\lambda,X)$ satisfies*

$$\Delta^\circ(m)\psi(\lambda,X)=-\langle\lambda,\lambda\rangle\psi(\lambda,X). \tag{2.4}$$

*The limit $\psi(\lambda,X)$ is the so-called Harish Chandra-Itzykson-Zuber (HIZ)-type integral, and it is well-known that it corresponds to spherical functions, or the Bessel functions, on the flat symmetric space $\mathfrak{p}$.*

***Remark* 2.4.** (i) Since $\varphi_\lambda=\varphi_\mu$ if and only if $\lambda=\omega\mu$ for $\omega\in\mathscr{W}_\Pi$, the same assertion holds for $\psi(\lambda,X)$.

(ii) The spherical functions $\psi(\lambda,X)$ are symmetric with respect to $\lambda$ and $X$, which is not obvious if one considers the system of differential equation (2.4) alone.

(iii) The contraction principle was used earlier in [D-R] for understanding the relationship between the principal series representations of $K\ltimes\mathfrak{p}$ and $G$. The limit approach was also used in [Ø-Z] to define the Weyl transform on flat symmetric spaces where $G/K$ is a Hermitian symmetric space. Another application of the limit approach can be found in [Cl] and [Sta].

It is remarkable that in spite of many results about the analysis of spherical functions on $G/K$, their Fourier analysis and asymptotic properties, it is only for very few cases that explicit formulas exist for these functions. From Theorem 2.3, one can see that for flat symmetric spaces we may derive at least the same amount of explicit information by a limit analysis, when the curvature goes to zero, as for spherical functions; and in some cases even more information is attained. In some interesting cases, for instance $SU^*(2n)/Sp(n)$ and $SU(p,q)/S(U(p)\times U(q))$, we were able to give in [B-Ø1] explicit formulas for the spherical functions $\psi(\lambda, X)$. In particular, these formulas give concrete solutions for problems of many body systems, which are related to quantum mechanics. We refer to [B-Ø1] for further details. Other interesting cases are also investigated. After [B-Ø1] was completed, we were able to give in [B-Ø2] a unified formula for the Bessel functions on flat symmetric spaces when $m_\alpha \in 2\mathbb{N}$ for all $\alpha \in \Sigma$. See Theorem 3.7, Table I, and Table II below for more information.

***Example* 2.5.** (The real rank-one case) This case corresponds to Riemannian symmetric spaces $G/K$ of noncompact type for which $\mathfrak{a}$ is one dimensional. There are only four type of groups $G$ with real rank-one, namely $SO_0(n,1), SU(n,1), Sp(n,1)$, and $F_{4(-20)}$.
The set $\Sigma^+$ consists at most of two elements, $\alpha$ and possibly $2\alpha$. By fixing $\alpha = 1$, we may identify $\mathfrak{a}$ and $\mathfrak{a}^*$ with $\mathbb{R}$, and their complexifications $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}$.
For symmetric spaces of real rank-one, the algebra $\mathscr{D}(G/K)$ of $G$-invariant differential operators on $G/K$ is generated by the Laplace-Beltrami operator, and the system of differential equation (2.2) is equivalent to the single Jacobi differential equation

$$\left\{\frac{d^2}{dt^2} + (m_\alpha \coth t + 2m_{2\alpha}\coth 2t)\frac{d}{dt}\right\} y = -(\lambda^2+\rho^2)y \qquad (t\in\mathbb{R}), \tag{2.5}$$

with $\lambda\in\mathbb{C}$ and $\rho = \frac{1}{2}(m_\alpha + 2m_{2\alpha})$. The spherical function

$$\varphi_\lambda(t) = {}_2F_1\left(\frac{i\lambda+\rho}{2}, \frac{-i\lambda+\rho}{2}; \frac{m_\alpha+m_{2\alpha}+1}{2}; -\operatorname{sh}^2 t\right)$$

is the unique solution to (2.5) which is even and satisfies $\varphi_\lambda(\mathbf{e}) = 1$ in the unit $\mathbf{e}$ of $G$. Here $m_\alpha = n-1$ and $m_{2\alpha} = 0$ for $G = SO_0(n,1)$, $m_\alpha = 2(n-1)$ and $m_{2\alpha} = 1$ for $G = SU(n,1)$, $m_\alpha = 4(n-1)$ and $m_{2\alpha} = 3$ for $G = Sp(n,1)$, and finally $m_\alpha = 8$ and $m_{2\alpha} = 7$ for $G = F_{4(-20)}$. Using the following well-known fact

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\Big\{1 + \frac{1}{2z}(a-b)(a+b-1) + \mathscr{O}(z^{-2})\Big\}, \quad \text{if } z\to\infty,$$

one can prove that

$$\psi(\lambda,t) = \Gamma\Big(\frac{m_\alpha+m_{2\alpha}+1}{2}\Big)\Big(\frac{\lambda t}{2}\Big)^{-\frac{m_\alpha+m_{2\alpha}-1}{2}} J_{\frac{m_\alpha+m_{2\alpha}-1}{2}}(\lambda t),$$

where $J_\nu$ is the Bessel function of the first kind.

***Example* 2.6.** (The complex case) Let $G$ be a Lie group with complex structure. Complex symmetric spaces are mainly characterized by the fact that $\Sigma$ is reduced and $m_\alpha = 2$ for all $\alpha\in\Sigma$. For the complex case, an explicit formula for the spherical functions $\varphi_\lambda$ on $G/K$ was given by Harish-Chandra, namely

$$\varphi_\lambda(\exp(X)) = \frac{\displaystyle\prod_{\alpha\in\Sigma^+}\langle\alpha,\rho\rangle \sum_{\omega\in\mathscr{W}_\Pi}(\det\omega)e^{\langle i\omega\lambda, X\rangle}}{\displaystyle\prod_{\alpha\in\Sigma^+}\langle\alpha,i\lambda\rangle \sum_{\omega\in\mathscr{W}_\Pi}(\det\omega)e^{\langle\omega\rho, X\rangle}}, \qquad X\in\mathfrak{a},$$

(cf. [Hel]). Notice that

$$\sum_{\omega\in\mathscr{W}_\Pi}(\det\omega)e^{\langle\omega\rho,X\rangle}=e^{\langle\rho,X\rangle}\prod_{\alpha\in\Sigma^+}(1-e^{-2\langle\alpha,X\rangle}).$$

Using Theorem 2.3, we obtain

$$\int_K e^{iB(A_\lambda,\mathrm{Ad}(k)X)}\,dk=\frac{\prod_{\alpha\in\Sigma^+}\langle\alpha,\rho\rangle}{\prod_{\alpha\in\Sigma^+}\langle\alpha,i\lambda\rangle\prod_{\alpha\in\Sigma^+}2\langle\alpha,X\rangle}\sum_{\omega\in\mathscr{W}_\Pi}(\det\omega)e^{\langle i\omega\lambda,X\rangle}. \tag{2.6}$$

In this case, i.e. when $G$ is complex, the integral representation of $\psi$ is also called Harish-Chandra integral, and its explicit expression (2.6) was proved earlier by Harish-Chandra in [HC2] using other techniques. Another proof is given by Berline, Getzler, and Vergne jointly in [Be-G-V] by using the orbit method. Our approach gives a new and simple proof of the Harish-Chandra integral.

We close this section by giving the Taylor expansion of $\psi(\lambda,X)$ in a series of Jack polynomials, when $G/K$ admits a root system of type $A_{N-1}$ ($N=2,3,\ldots$). This follows from Theorem 2.3 by using the generalized binomial formula of the spherical functions on $G/K$ proved in [Ok-Ol1]. After this paper was completed, the referee pointed out for us the article [Ok-Ol2] where the Taylor expansion was also proved.

Consider the following list of noncompact symmetric spaces with root system of type $A_{N-1}$ ($N=2,3,\dots$)

$$GL(N,\mathbb{R})/O(N),\quad GL(N,\mathbb{C})/U(N),\quad GL(N,\mathbb{H})/Sp(N),$$
$$E_{6(-26)}/F_4,\quad O(1,N)/O(N).$$

We use throughout this section the notation related to Jack polynomials from [S-V]. Let $S_N$ be the symmetric group of degree $N$. Let $\mathbb{C}[x]$ be the algebra of polynomials in $N$ variables $x=(x_1,\ldots,x_N)$, with coefficients in $\mathbb{C}$. Set $\mathbb{C}[x]^{S_N}$ to be the algebra of symmetric polynomials in $\mathbb{C}[x]$. A partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_N,\ldots)$ is a sequence of finitely many nonzero integers, such that $\lambda_i\geq 0$ and $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_N\geq\cdots$. If $\lambda_1+\cdots+\lambda_N+\cdots=m$, then we write $|\lambda|=m$. Further, we denote by $\ell(\lambda)$ the number of nonzero terms in $\lambda$. Finally for two partitions $\mu$ and $\lambda$, we write $\mu\leq\lambda$ if $|\mu|=|\lambda|$ and

$$\mu_1+\mu_2+\cdots+\mu_i\leq\lambda_1+\lambda_2+\cdots+\lambda_i,$$

for all $i\geq 1$.

Let $\wp$ be a strictly positive parameter and consider the following Calogero-Moser-Sutherland operator

$$\Delta_\wp^N=\sum_{i=1}^N\left(x_i\frac{\partial}{\partial x_i}\right)^2+2\wp\sum_{i\neq j}\frac{x_ix_j}{x_i-x_j}\frac{\partial}{\partial x_i}.$$

For a partition $\lambda$ such that $\ell(\lambda)\leq N$, Jack's polynomials $P_\lambda(x,\wp)\in\mathbb{C}[x]^{S_N}$ are uniquely specified by

- $P_\lambda(x,\wp)$ is an eigenfunction of the $\Delta_\wp^N$ operator,
- $P_\lambda(x,\wp)=m_\lambda+\sum_{\mu\leq\lambda}v_{\lambda,\mu}(\wp)m_\mu$, where $v_{\lambda,\mu}(\wp)\in\mathbb{C}$ and $m_\mu$ is the elementary symmetric polynomial.

By [M] and [St], we have the following branching rule for the Jack polynomials

$$P_\lambda(x_1,x_2,\ldots,x_N,\wp)=\sum_{\mu\prec\lambda}\varphi_{\lambda/\mu}(\wp)x_1^{|\lambda/\mu|}P_\mu(x_2,\ldots,x_N;\wp), \tag{2.7}$$

where $\mu\prec\lambda$ stands for the inequalities of interlacing

$$\lambda_1\geq\mu_1\geq\lambda_2\geq\mu_2\geq\cdots\geq\mu_{N-1}\geq\lambda_N,$$

the weight of the skew diagram $\lambda/\mu$ equals $|\lambda|-|\mu|$, and $\varphi_{\lambda/\mu}(\wp)$ is the following coefficient

$$\varphi_{\lambda/\mu}(\wp)=\prod_{1\leq i\leq j\leq N-1}\frac{(\mu_i-\mu_j+\wp(j-i)+\wp)_{\mu_j-\lambda_{j+1}}}{(\mu_i-\mu_j+\wp(j-i)+1)_{\mu_j-\lambda_{j+1}}}\frac{(\lambda_i-\mu_j+\wp(j-i)+1)_{\mu_j-\lambda_{j+1}}}{(\lambda_i-\mu_j+\wp(j-i)+\wp)_{\mu_j-\lambda_{j+1}}}.$$

Next we shall recall the so-called shifted Jack polynomials (cf. [Kn-Sa], [Ok-Ol1]). Denote by $\mathbb{C}[x]_\wp^{S_N}$ the algebra of polynomials which are symmetric in the $N$ shifted variables $x_i+(1-i)\wp$. Henceforth, we will identify a partition $\lambda$ with its diagram

$$\lambda\equiv\left\{\ \square=(i,j)\mid 1\leq i\leq \ell(\lambda),\ 1\leq j\leq\lambda_i\ \right\}.$$

Thus, for a partition $\lambda$ set

$$H(\lambda,\wp)=\prod_{\square\in\lambda}(c_\wp(\square)+1),$$

where

$$c_\wp(\square)=\lambda_i-j+\wp(\lambda'_j-i).$$

The shifted Jack polynomials $P^*_\lambda(x,\wp)\in\mathbb{C}[x]_\wp^{S_N}$ are uniquely specified by $\deg(P^*_\lambda)\leq|\lambda|$, and

$$P^*_\lambda(\mu,\wp)=\begin{cases}H(\lambda,\wp), & \mu=\lambda;\\ 0, & |\mu|\leq|\lambda|,\quad \mu\neq\lambda.\end{cases}$$

Here we assume that $\ell(\lambda)\leq N$. Further, in [Kn-Sa] the authors proved that $P^*_\lambda(\mu,\wp)=0$ unless $\mu\subset\lambda$, i.e. $\lambda_i-\mu_i$ is positive for every $i$, and that $P^*_\lambda(x,\wp)$ coincides with the Jack polynomial $P_\lambda(x,\wp)$ plus lower degree terms.

In [Ok], the following branching formula was proved

$$P^*_\lambda(x_1,\dots,x_N;\wp)=\sum_{\mu\prec\lambda}\varphi_{\lambda/\mu}(\wp)\prod_{\square\in\lambda/\mu}(x_1-c'_\wp(\square))P^*_\mu(x_2,\dots,x_N;\wp),\tag{2.8}$$

where $\varphi_{\lambda/\mu}(\wp)$ is the same as for $P_\lambda$, and

$$c'_\wp(\square)=(j-1)-\wp(i-1),\qquad \square=(i,j).$$

Next, we review the spherical functions on symmetric cones. For details, we refer to Faraut-Korányi's book [F-K].

Assume that $\mathbb{V}$ is a simple Euclidean Jordan algebra, i.e. $\mathbb{V}$ does not contain non-trivial ideals, and let $\Omega$ be an open and convex cone associated to $\mathbb{V}$. The Riemannian symmetric space $\Omega$ can be identified with one of the five symmetric spaces $G/K$ listed above. Set $N$ to be the rank of $\mathbb{V}$, and let $\{c_1,\dots,c_N\}$ be a complete system of orthogonal idempotent elements. Each element $x$ in $\mathbb{V}$ can be written as $x=k\sum_{i=1}^N x_ic_i$, with $k\in K$ and $x_i\in\mathbb{R}$.

For $\mathbf{m}=(m_1,\dots,m_N)$ and $x=\sum_{j=1}^N x_jc_j$, the spherical functions on $\Omega$ are given by

$$\varphi_{\mathbf{m}}(x)=\int_K\Delta_1^{m_1-m_2}(kx)\cdots\Delta_{N-1}^{m_{N-1}-m_N}(kx)\Delta_N^{m_N}(kx)\,dk,$$

where $\Delta_j(y)$ is the principal minor of order $j$ of $y$, and $dk$ denotes the normalized Haar measure on $K$. This formula corresponds to the classical Harish-Chandra formula for the spherical functions on $G/K$ with $\mathbf{m}=\frac{\lambda+\rho}{2}$, where $\rho=(\rho_1,\dots,\rho_N)$, $\rho_j=\frac{1}{2}(2j-N-1)$, and $\lambda\in\mathbb{C}^N$.

If $\mathbf{m}=(m_1,\dots,m_N)\in\mathbb{N}^N$ such that $m_1\geq\cdots\geq m_N\geq 0$, then the spherical function $\varphi_{\mathbf{m}}$ is a polynomial, and can be written in terms of the Jack polynomials. If $x=k\sum_{j=1}^N x_jc_j$,

$$\varphi_{\mathbf{m}}(x)=\frac{P_{\mathbf{m}}(x_1,\dots,x_N;\wp)}{P_{\mathbf{m}}(1,\dots,1;\wp)},\qquad\text{with }\wp=\frac{2}{d},$$

where

$$\begin{cases} d=1 & \text{for} \quad GL(N,\mathbb{R})/O(N), \\ d=2 & \text{for} \quad GL(N,\mathbb{C})/U(N), \\ d=4 & \text{for} \quad GL(N,\mathbb{H})/Sp(N), \\ d=8 & \text{for} \quad E_{6(-26)}/F_4, \quad \text{and} \\ d=N & \text{for} \quad O(1,N)/O(N). \end{cases}$$

By [Ok-Ol1, (2.6)], we have the following binomial formula

$$\frac{P_{\mathbf{m}}(1+x_1,\dots,1+x_N;\wp)}{P_{\mathbf{m}}(1,\dots,1;\wp)} = \sum_{\mu\subset\mathbf{m}} \frac{P^*_\mu(\mathbf{m},\wp)P_\mu(x,\wp)}{P_\mu(1,\dots,1;\wp)H(\mu,\wp)}. \tag{2.9}$$

(Recall that $\mu\subset\mathbf{m}$ stands for $\mu_i\le m_i$ for all $i$.) For $\wp=1$, formula (2.9) reduces to the usual binomial formula

$$\frac{S_{\mathbf{m}}(1+x_1,\dots,1+x_N)}{S_{\mathbf{m}}(1,\dots,1)} = \sum_{\mu\subset\mathbf{m}} \frac{S^*_\mu(\mathbf{m})S_\mu(x)}{\prod_{\square=(i,j)\in\mu}(N+j-i)},$$

where $S_\mu$ is the Schur function

$$S_\mu(x) = \frac{\det\left(x_i^{\mu_j+N-j}\right)_{1\le i,j\le N}}{\prod_{1\le i<j\le N}(x_i-x_j)},$$

and

$$S^*_\mu(\mathbf{m}) = \frac{\det\big((m_i+N-i)\cdots(m_i-i+j-\mu_j+1)\big)_{1\le i,j\le N}}{\prod_{1\le i<j\le N}(m_i-i-m_j+j)}.$$

Henceforth, $\varsigma$ denotes a very large integer. After using formula (2.7) $N$-times, we obtain

$$P_\mu(1-e^{s_1/\varsigma},\dots,1-e^{s_N/\varsigma};\wp)\sim\varsigma^{-|\mu|}P_\mu(s_1,\dots,s_N;\wp) \quad \text{as } \varsigma\to\infty.$$

The same argument for the shifted Jack polynomials gives

$$P^*_\mu(m_1\varsigma,\dots,m_N\varsigma;\wp)\sim\varsigma^{|\mu|}P_\mu(m_1,\dots,m_N;\wp), \quad \text{as } \varsigma\to\infty.$$

Thus, by (2.9) the following holds.

**Theorem 2.7.** (cf. [Ok-Ol2], [B-Ø1]) *For $X=\sum_{i=1}^N x_ic_i$, $\mathbf{m}=(m_1,\dots,m_N)\in\mathbb{N}^N$ such that $m_1\ge\cdots\ge m_N$, and $\varsigma\in\mathbb{N}$, the following Taylor series holds*

$$\lim_{\varsigma\to\infty}\varphi_{\mathbf{m}\varsigma}\big(\exp(\varsigma^{-1}X)\big) = \sum_{\mu\in\mathscr{P}} \frac{P_\mu(m_1,\dots,m_N;\frac{2}{d})P_\mu(x_1,\dots,x_N;\frac{2}{d})}{P_\mu(1,\dots,1;\frac{2}{d})H(\mu;\frac{2}{d})}.$$

***Remark* 2.8.** (i) For a real vector $\nu=(\nu_1,\dots,\nu_N)$ such that $\nu_1\ge\cdots\ge\nu_N$, write

$$\psi_c(\nu,X) := \int_K e^{B(A_\nu,\mathrm{Ad}(k)X)}\,dk.$$

As in Theorem 2.3, one can prove that

$$\begin{aligned}\psi_c(\nu,X) &= \lim_{\varsigma\to\infty}\varphi_{[\varsigma\nu]}\big(\exp(\varsigma^{-1}X)\big)\\ &= \sum_{\mu\in\mathscr{P}} \frac{P_\mu(\nu_1,\dots,\nu_N;\frac{2}{d})P_\mu(x_1,\dots,x_N;\frac{2}{d})}{P_\mu(1,\dots,1;\frac{2}{d})H(\mu;\frac{2}{d})}, \qquad \text{(by Theorem 2.7)}\end{aligned}$$

where $[\varsigma\nu]=([\varsigma\nu_1],\dots,[\varsigma\nu_N])$ is the $N$-vector of integral parts.

(ii) Using [Sw], one can prove that for $\mathbf{m}=(m_1,m_2)$,

$$\varphi_{\mathbf{m}}(x_1,x_2) = e^{m_1x_1}e^{m_2x_2}{}_2F_1\Big(m_2-m_1,\frac{d}{2};d;1-e^{x_2-x_1}\Big).$$

Hence, if $N=2$ we may rewrite the Bessel function $\psi_c(\nu_1,\nu_2;x_1,x_2)$ as

$$\psi_c(\nu_1,\nu_2;x_1,x_2)=e^{\nu_1x_1}e^{\nu_2x_2}{}_1F_1\Big(\frac{d}{2},d;(\nu_1-\nu_2)(x_1-x_2)\Big),$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind.

## 3. Bessel functions related to root systems

As stated in Example 2.5, the spherical functions $\varphi_\lambda$ on Riemannian symmetric spaces $G/K$ of rank-one are a special type of hypergeometric functions. The specialization comes from the choice of the multiplicities $m_\alpha$ and $m_{2\alpha}$ as the dimension of the root spaces $\mathfrak{g}^{(\alpha)}$ and $\mathfrak{g}^{(2\alpha)}$, respectively. This specialization is still effective even for symmetric spaces with higher-rank by constraining the root multiplicities $m_\alpha$ to assume particular positive integer values. The spherical functions are determined as well by the geometry, since $\varphi_\lambda$ is an eigenfunction of each $G$-invariant differential operator that belongs to $\mathscr{D}(G/K)$. However, the differential system (2.2) still makes sense without the geometric restriction on the multiplicities $m_\alpha$. This was the starting point of Heckman-Opdam's theory on hypergeometric functions. Their objective was to generalize Harish-Chandra's theory of spherical functions for arbitrary complex values of multiplicities associated with root systems (cf. [He-O], [O1], [O2]).

In Heckman-Opdam's theory, the Riemannian symmetric spaces $G/K$ are replaced by the following ingredients: an $N$-dimensional real Euclidean vector space $\mathfrak{a}$ with fixed inner product $\langle\cdot,\cdot\rangle$, a root system $\mathscr{R}$ in the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ – which is assumed to satisfy the crystallographic condition, i.e. $2\langle\alpha,\beta\rangle/\langle\beta,\beta\rangle\in\mathbb{Z}$ for all $\alpha,\beta\in\mathscr{R}$ –, and finally a multiplicity function $k:\mathscr{R}\to\mathbb{C}$ invariant under the action of the Weyl group associated with $\mathscr{R}$.

The symbols $\mathfrak{a}_{\mathbb{C}},\mathfrak{a}_{\mathbb{C}}^*,\mathfrak{a}^+,A,A_{\mathbb{C}}$, and $A^+$ will have the same meaning as in the previous section.

Let $\mathscr{R}^+$ be a choice of positive roots in $\mathscr{R}$, and denote by $\Pi$ the corresponding fundamental system of simple roots. The Weyl group $\mathscr{W}_\Pi$ is generated by the reflections $r_\alpha:\mathfrak{a}^*\to\mathfrak{a}^*$, with $\alpha\in\Pi$ and $r_\alpha(\lambda)=\lambda-\lambda(\breve{\alpha})\alpha\in\mathfrak{a}^*$. Here $\lambda(\breve{\alpha}):=2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle$. The action of $\mathscr{W}_\Pi$ extends to $\mathfrak{a}$ by duality, to $\mathfrak{a}_{\mathbb{C}}^*$ and $\mathfrak{a}_{\mathbb{C}}$ by $\mathbb{C}$-linearity, and to $A$ and $A_{\mathbb{C}}$ by the exponential map.

Let $k:\mathscr{R}\to\mathbb{C}$ be a multiplicity function. Setting $k_\alpha:=k(\alpha)$ for $\alpha\in\mathscr{R}$, we have $k_{w\alpha}=k_\alpha$ for all $w\in\mathscr{W}_\Pi$. Denote by $\mathscr{K}$ the set of all multiplicity functions on $\mathscr{R}$. If $m:=\sharp\{\mathscr{W}_\Pi\text{−orbits in }\mathscr{R}\}$, then $\mathscr{K}\cong\mathbb{C}^m$.

The set $A_{\mathbb{C}}^{\rm reg}:=\big\{a\in A_{\mathbb{C}}\mid e^{\alpha(\log a)}\neq1\ \forall\,\alpha\in\mathscr{R}\big\}$ consists of the regular elements of $A_{\mathbb{C}}$ for the $\mathscr{W}_\Pi$-action. Notice that $A^+$ is a subset of $A_{\mathbb{C}}^{\rm reg}$. Denote by $\mathbb{C}[A_{\mathbb{C}}^{\rm reg}]$ the algebra of regular functions on $A_{\mathbb{C}}^{\rm reg}$.

Let $S(\mathfrak{a}_{\mathbb{C}})$ be the symmetric algebra over $\mathfrak{a}_{\mathbb{C}}$ considered as the space of polynomial functions on $\mathfrak{a}_{\mathbb{C}}$, and let $S(\mathfrak{a}_{\mathbb{C}})^{\mathscr{W}_\Pi}$ be the subalgebra of $\mathscr{W}_\Pi$-invariant elements. Each $p\in S(\mathfrak{a}_{\mathbb{C}})$ defines a constant-coefficient differential operator $\partial(p)$ on $A_{\mathbb{C}}$, and on $\mathfrak{a}_{\mathbb{C}}$, such that, for all $H\in\mathfrak{a}$, $\partial(H)$ is the directional derivative in the direction of $H$. The algebra of the differential operators $\partial(p)$, with $p\in S(\mathfrak{a}_{\mathbb{C}})$, will also be denoted by $S(\mathfrak{a}_{\mathbb{C}})$. Let $\mathscr{D}(A_{\mathbb{C}}^{\rm reg}):=\mathbb{C}[A_{\mathbb{C}}^{\rm reg}]\otimes S(\mathfrak{a}_{\mathbb{C}})$ be the algebra of differential operators on $A_{\mathbb{C}}$ with coefficients in $\mathbb{C}[A_{\mathbb{C}}^{\rm reg}]$. The Weyl group action on $\mathscr{D}(A_{\mathbb{C}}^{\rm reg})$ is given by

$$w(f\otimes\partial(p))=wf\otimes\partial(wp),\qquad w\in\mathscr{W}_\Pi.$$

Set $\mathscr{D}(A_{\mathbb{C}}^{\rm reg})^{\mathscr{W}_\Pi}$ to be the subspace of $\mathscr{W}_\Pi$-invariant elements.

For $k\in\mathscr{K}$ and for a fixed orthonormal basis $\{\xi_i\}_{i=1}^N$ of $\mathfrak{a}$, write

$$\Delta(k):=\sum_{j=1}^N\partial_{\xi_j}^2+\sum_{\alpha\in\mathscr{R}^+}k_\alpha\coth\Big(\frac{\alpha}{2}\Big)\partial_\alpha. \tag{3.1}$$

Here we write $\partial_{\xi_i}$ for $\partial(\xi_i)$. Denote by $\widetilde{\mathscr{D}}(A_{\mathbb{C}}^{\mathrm{reg}})$ the commutator of $\Delta(k)$ in $\mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}})^{\mathscr{W}_\Pi}$. In [He-O], Heckman and Opdam proved that $\widetilde{\mathscr{D}}(A_{\mathbb{C}}^{\mathrm{reg}})$ is a commutative algebra, parameterized by the elements of $S(\mathfrak{a}_{\mathbb{C}})^{\mathscr{W}_\Pi}$. In [C], Cherednik was able to give an algebraic algorithm for constructing the operators $D(k,p) \in \widetilde{\mathscr{D}}(A_{\mathbb{C}}^{\mathrm{reg}})$ corresponding to $p \in S(A_{\mathbb{C}})^{\mathscr{W}_\Pi}$ by using the so-called Dunkl-Cherednik operators. We recall briefly this algorithm following [He2]. For $k \in \mathscr{K}$ and $\xi \in \mathfrak{a}_{\mathbb{C}}$, the Dunkl-Cherednik operators $T(\xi,k) \in \mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[\mathscr{W}_\Pi]$ are defined by

$$T(\xi,k) := \partial_\xi - \rho(k)(\xi) + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \alpha(\xi)(1-e^{-\alpha})^{-1} \otimes (1-r_\alpha), \tag{3.2}$$

where $\rho(k) := (1/2)\sum_{\alpha\in\mathscr{R}^+} k_\alpha \alpha \in \mathfrak{a}_{\mathbb{C}}^*$. In particular, for all $\xi,\eta \in \mathfrak{a}_{\mathbb{C}}$ and $k \in \mathscr{K}$, we have $T(\xi,k)T(\eta,k) = T(\eta,k)T(\xi,k)$. Due to the latter fact, the map $\mathfrak{a}_{\mathbb{C}} \to \mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[\mathscr{W}_\Pi]$, $\xi \mapsto T(\xi,k)$, can be extended in a unique way to an algebra homomorphism $S(\mathfrak{a}_{\mathbb{C}}) \to \mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[\mathscr{W}_\Pi]$. The image of $p \in S(\mathfrak{a}_{\mathbb{C}})$ will be denoted by $T(p,k)$. Suppose $p \in S(\mathfrak{a}_{\mathbb{C}})^{\mathscr{W}_\Pi}$, then by [He2]

$$T(p,k) = \sum_{w\in\mathscr{W}_\Pi} D(w,p,k) \otimes w \in \mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[\mathscr{W}_\Pi].$$

Moreover, if we denote by "Proj" the map from $\mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}}) \otimes \mathbb{C}[\mathscr{W}_\Pi]$ to $\mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}})$ given by $\mathrm{Proj}(\sum_i D_i \otimes w) = \sum_i D_i$, then

$$D(p,k) := \mathrm{Proj}(T(p,k)) = \sum_{w\in\mathscr{W}_\Pi} D(w,p,k) \in \mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}})^{\mathscr{W}_\Pi}.$$

The operator $D(p,k)$ is the unique element in $\mathscr{D}(A_{\mathbb{C}}^{\mathrm{reg}})^{\mathscr{W}_\Pi}$ which has the same restriction to $\mathbb{C}[A_{\mathbb{C}}]^{\mathscr{W}_\Pi}$ as $T(p,k)$. By [He2] the element $D(p,k)$ preserves $\mathbb{C}[A_{\mathbb{C}}]^{\mathscr{W}_\Pi}$, and $D(p,k)D(q,k) = D(pq,k)$ for $p,q \in S(\mathfrak{a}_{\mathbb{C}})^{\mathscr{W}_\Pi}$. Thus the set $\{D(p,k) \mid p \in S(\mathfrak{a}_{\mathbb{C}})^{\mathscr{W}_\Pi}\}$ is a commutative algebra of differential operators. For instance, if $p_0 = \sum_{j=1}^N \xi_j^2$, where $\{\xi_j\}_{j=1}^N$ is the fixed orthonormal basis of $\mathfrak{a}$, then

$$D(p_0,k) = \Delta(k) + \langle \rho(k), \rho(k)\rangle,$$

where $\Delta(k)$ is the Laplacian operator (3.1).

***Example* 3.1.** Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$, and $\Sigma(\mathfrak{g},\mathfrak{a})$ be the restricted root system associated with $\mathfrak{a}$. If we put $\mathscr{R} := 2\Sigma(\mathfrak{g},\mathfrak{a})$ and $k_{2\alpha} := \frac{1}{2}m_\alpha$, where $m_\alpha$ is the multiplicity of the root $\alpha$, then $\Delta(k)$ coincides with the radial part of the Laplace-Beltrami operator on the symmetric space $G/K$. The set $\widetilde{\mathscr{D}}(A_{\mathbb{C}}^{\mathrm{reg}})$ represents the commutative algebra of radial parts on $A^+$ of the differential operators in $\mathscr{D}(G/K)$. This setting corresponds to the situation in the previous section which we shall refer to by the geometric case.

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the following system of differential equations

$$D(p,k)F = p(\lambda)F, \qquad p \in S(\mathfrak{a}_{\mathbb{C}})^{\mathscr{W}_\Pi}, \tag{3.3}$$

is the so-called hypergeometric system of differential equations associated with the root system $\mathscr{R}$. In particular, if $p_0 := \sum_{j=1}^N \xi_j^2$, the differential equations (3.3) become

$$\Delta(k)F = \big(\langle\lambda,\lambda\rangle - \langle\rho(k),\rho(k)\rangle\big)F. \tag{3.4}$$

In the geometric case, the hypergeometric system (3.3) coincides with the system of differential equations (2.1) defining Harish-Chandra's spherical functions $\varphi_{i\lambda}$.

By the explicit expression of the differential equation (3.4), Heckman and Opdam searched for solutions for the hypergeometric system on $A^+ = \exp(\mathfrak{a}^+)$ of the form

$$\Phi(\lambda,k,a) = \sum_{\ell\geq 0} \Gamma_\ell(\lambda,k) e^{\lambda-\rho(k)-\ell(\log a)}, \qquad a \in A^+, \tag{3.5}$$

where $\Gamma_0(\lambda,k)=1$ and $\Gamma_\ell(\lambda,k)\in\mathbb{C}$ satisfying some recurrence relations [He-O]. Using $\Phi(\lambda,k,\cdot)$, Heckman and Opdam were able to build a basis for the solution space of the entire hypergeometric system with spectral parameter $\lambda$. This is possible if $\lambda$ is generic, i.e. $\lambda(\breve{\alpha})\notin\mathbb{Z}$ for all $\alpha\in\mathscr{R}$. To write the main result of Heckman and Opdam, set

$$\widetilde{c}(\lambda,k):=\prod_{\alpha\in\mathscr{R}^+}\frac{\Gamma\big(\lambda(\breve{\alpha})+\frac{1}{2}k_{\frac{\alpha}{2}}\big)}{\Gamma\big(\lambda(\breve{\alpha})+\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha\big)},\qquad (\lambda,k)\in\mathfrak{a}^*_{\mathbb{C}}\times\mathscr{K},\tag{3.6}$$

and define the following meromorphic $c$-function

$$c(\lambda,k):=\frac{\widetilde{c}(\lambda,k)}{\widetilde{c}(\rho(k),k)}.$$

**Theorem 3.2.** (cf. [He-O], [He-S]) *Let $S:=\big\{$zeros of the entire function $\tilde{c}(\rho(k),k)\big\}$. There exists a $\mathscr{W}_\Pi$-invariant tubular neighborhood $U$ of $A$ in $A_{\mathbb{C}}$ such that the hypergeometric function*

$$F(\lambda,k,a):=\sum_{w\in\mathscr{W}_\Pi}c(w\lambda,k)\Phi(w\lambda,k,a)$$

*is a holomorphic function on $\mathfrak{a}^*_{\mathbb{C}}\times(\mathscr{K}\setminus S)\times U$. Moreover*

$$F(w\lambda,k,a)=F(\lambda,k,wa)=F(\lambda,k,a),$$

*for all $w\in\mathscr{W}_\Pi$ and $(\lambda,k,a)\in\mathfrak{a}^*_{\mathbb{C}}\times(\mathscr{K}\setminus S)\times U$.*

The functions $F(\lambda,k,a)$ are nowadays known as Heckman-Opdam hypergeometric functions.

To obtain the Bessel functions related to the root system $\mathscr{R}$ as an appropriate limit of Heckman-Opdam hypergeometric functions, we will proceed as in Section 2.

For strictly positive small real $\varepsilon$, we introduce the following procedure

$$(\mathscr{P})\quad\begin{array}{l}\text{(i) substitute }\alpha\text{ by }\varepsilon\alpha,\\ \text{(ii) substitute }\xi\text{ by }\varepsilon^{-1}\xi,\text{ and}\\ \text{(iii) set }k_{\epsilon\alpha}:=k_\alpha.\end{array}$$

Applying the procedure $(\mathscr{P})$ to the definition of the operator $T(\xi,k)$ in (3.2), we obtain

$$T^{(\varepsilon)}(\xi,k)\;=\;\frac{1}{\varepsilon}\partial_\xi-\rho(k)(\xi)+\sum_{\alpha\in\mathscr{R}^+}k_\alpha\frac{\alpha(\xi)}{\varepsilon\alpha}\sum_{m=0}^{\infty}\frac{B_m\varepsilon^m\alpha^m}{m!}(1-r_\alpha),$$

where $B_m$ is the $m$-th Bernoulli number. Clearly the following limit exists

$$\lim_{\varepsilon\to0}\varepsilon T^{(\varepsilon)}(\xi,k)=T^\circ(\xi,k),$$

where

$$T^\circ(\xi,k)=\partial_\xi+\sum_{\alpha\in\mathscr{R}^+}k_\alpha\frac{\alpha(\xi)}{\alpha}(1-r_\alpha).\tag{3.7}$$

The differential operator $T^\circ(\xi,k)$ is the so-called Dunkl operator [Du]. We are not aware of any previous work mentioning a connection of this type between the Dunkl-Cherednik operators $T(\xi,k)$ and the Dunkl operators $T^\circ(\xi,k)$. Another type of relation between $T(\xi,k)$ and $T^\circ(\xi,k)$ can be found in [T].

Further, by the procedure $(\mathscr{P})$, the operator $D(p_0,k)=\mathrm{Proj}(T(p_0,k))=\Delta(k)+\langle\rho(k),\rho(k)\rangle$ becomes $\Delta^{(\varepsilon)}(k)+\langle\rho(k),\rho(k)\rangle$, where

$$\Delta^{(\varepsilon)}(k)=\frac{1}{\varepsilon^2}\sum_{j=1}^{N}\partial^2_{\xi_j}+\frac{1}{\varepsilon^2}\sum_{\alpha\in\mathscr{R}^+}\frac{2k_\alpha}{\alpha}\partial_\alpha+\sum_{\alpha\in\mathscr{R}^+}\sum_{m=1}^{\infty}\frac{k_\alpha B_{2m}}{(2m)!}\left(\frac{\alpha}{2}\right)^{2m-1}\varepsilon^{2(m-1)}\partial_\alpha,$$

and $B_{2m}$ is the $2m$-th Bernoulli number. Thus

$$\lim_{\varepsilon\to 0}\varepsilon^2\big[\Delta^{(\varepsilon)}(k)+\langle\rho(k),\rho(k)\rangle\big]=\sum_{j=1}^{N}\partial_{\xi_j}^2+\sum_{\alpha\in\mathscr{R}^+}k_\alpha\Big(\frac{2}{\alpha}\Big)\partial_\alpha. \tag{3.8}$$

In the geometric case, this is equivalent to Theorem 2.2. We will denote by $\Delta^\circ(k)$ the right-hand side of (3.8).

To derive the eigenfunctions of $\Delta^\circ(k)$ by a limit analysis of Heckman-Opdam hypergeometric functions, we will use the same argument to that in the previous section for Harish-Chandra's spherical functions. We should notice that $F(\lambda,k,a)$ does not have an integral representation, since there is no longer a group theory behind. To get over this missing fact, we will proceed by induction on the multiplicity functions $k$.

From the definition of the $\tilde{c}$-function, one can see that $\tilde{c}(\lambda,0)=1$. By [He-S, (3.5.14)] $\lim_{k\to 0}\tilde{c}(\rho(k),k)=|\mathscr{W}_\Pi|$, and therefore

$$F(\lambda,0,a)=\frac{1}{|\mathscr{W}_\Pi|}\sum_{w\in\mathscr{W}_\Pi}e^{w\lambda(\log a)},\qquad a\in A.$$

In particular, the following limit formula holds

$$\lim_{\varepsilon\to 0}F\Big(\frac{\lambda}{\varepsilon},0,\exp(\varepsilon X)\Big)=\frac{1}{|\mathscr{W}_\Pi|}\sum_{w\in\mathscr{W}_\Pi}e^{w\lambda(X)},\quad X\in\mathfrak{a}. \tag{3.9}$$

Next, for $X\in\mathfrak{a}$ and $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$, we will denote by $F^\circ(\lambda,k,X)$ the limit of $F\big(\frac{\lambda}{\varepsilon},k,\exp(\varepsilon X)\big)$ as $\varepsilon\to 0$, if it exists.

Let $\mathscr{Z}$ be the set of $k\in\mathscr{K}$ such that $k_\alpha\in\mathbb{Z}$ for $\alpha\in\mathscr{R}\setminus\frac{1}{2}\mathscr{R}$, and $k_\alpha\in 2\mathbb{Z}$ for $\alpha\in\mathscr{R}\cap\frac{1}{2}\mathscr{R}$. Further, let $\mathscr{Z}^+$ denote the set of multiplicity functions $k\in\mathscr{K}$ such that (i) $k_\alpha\in\mathbb{Z}^+$ for $\alpha\in\mathscr{R}\setminus\frac{1}{2}\mathscr{R}$, (ii) $k_\alpha\in 2\mathbb{Z}^+$ for $\alpha\in\mathscr{R}\cap\frac{1}{2}\mathscr{R}$, and (iii) $\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha\in\mathbb{Z}^+$ with the convention $k_{\frac{\alpha}{2}}=0$ if $\frac{\alpha}{2}\notin\mathscr{R}$. We mention that the results in [Du-J-O, Section 4] show by inspection that integral $k$'s are not among the singular set $S$ introduced in Theorem 3.2.

The main tool in the induction process is to use the so-called Opdam shift operators $\mathbb{G}_\pm(\pm\ell,k)$ of shifts $\pm\ell$, where $\ell\in\mathscr{Z}^+$ and $k\in\mathscr{Z}$. These operators satisfy

$$\begin{aligned}\mathbb{G}_-(-\ell,k)\Phi(\lambda,k,a)&=\frac{\tilde{c}(\lambda,k-\ell)}{\tilde{c}(\lambda,k)}\Phi(\lambda,k-\ell,a),\\ \mathbb{G}_+(\ell,k)\Phi(\lambda,k,a)&=\frac{\tilde{c}(-\lambda,k)}{\tilde{c}(-\lambda,k+\ell)}\Phi(\lambda,k+\ell,a).\end{aligned}$$

We refer to [O1] and [O2] for more details on $\mathbb{G}_\pm(\pm\ell,k)$.

Using the explicit forms of $\mathbb{G}_\pm(\pm\ell,k)$ and the procedure $(\mathscr{P})$, we obtain two deformed operators $\mathbb{G}_\pm^{(\varepsilon)}(\pm\ell,k)$, and we prove that the following limits exist

$$\mathbb{G}_+^\circ(\ell,k):=\lim_{\varepsilon\to 0}\varepsilon^{2\sum_{\alpha>0}\ell_\alpha}\mathbb{G}_+^{(\varepsilon)}(\ell,k), \tag{3.10}$$

$$\mathbb{G}_-^\circ(-\ell,k):=\lim_{\varepsilon\to 0}\mathbb{G}_-^{(\varepsilon)}(-\ell,k). \tag{3.11}$$

We refer to [B-Ø2] for the explicit expressions of $\mathbb{G}_\pm^\circ(\pm\ell,k)$. Further results on $\mathbb{G}_\pm^\circ(\pm\ell,k)$ are also obtained. Note that when $\mathscr{R}$ is reduced, the shift operators $\mathbb{G}_\pm^\circ(\pm\ell,k)$ can also be constructed by composing fundamental shift operators of shift $\pm 1$.

Henceforth we shall restrict ourselves to reduced root systems.

Recall that $F^\circ(\lambda,k,X)$ exists for $k\equiv 0$. Using the shift operators $\mathbb{G}_\pm^\circ(\pm\ell,k)$, we prove the following theorem by induction on $k$.

**Theorem 3.3.** (cf. [B-Ø2]) *Assume that $\mathscr{R}$ is reduced. For all $k \in \mathscr{Z}$, the following limit and its derivatives exist*

$$\lim_{\varepsilon \to 0} F\Big(\frac{\lambda}{\varepsilon}, k, \exp(\varepsilon X)\Big) = F^\circ(\lambda, k, X),$$

*and it satisfies the following Bessel system of differential equations on $\mathscr{W}_\Pi \setminus \mathfrak{a}_\mathbb{C}$*

$$T^\circ(p, \xi)_{\big|\mathbb{C}[\mathfrak{a}_\mathbb{C}]^{\mathscr{W}_\Pi}} \psi = p(\lambda)\psi, \qquad \forall\, p \in S(\mathfrak{a}_\mathbb{C})^{\mathscr{W}_\Pi}. \tag{3.12}$$

*Moreover, for $\ell \in \mathscr{Z}^+$*

$$\mathbb{G}_+^\circ(\ell, k) F^\circ(\lambda, k, X) = \lambda^{2\sum_{\alpha\in\mathscr{R}^+}\ell_\alpha} \frac{\tilde{c}(\rho(k+\ell), k+\ell)}{\tilde{c}(\rho(k), k)} F^\circ(\lambda, k+\ell, X),$$

$$\mathbb{G}_-^\circ(-\ell, k) F^\circ(\lambda, k, X) = \frac{\tilde{c}(\rho(k-\ell), k-\ell)}{\tilde{c}(\rho(k), k)} F^\circ(\lambda, k-\ell, X).$$

In the light of Example 3.1, the above theorem contains Theorem 2.3 as a particular case only when $m_\alpha \in 2\mathbb{N}$. Note that $m_\alpha \in 2\mathbb{N}$ guarantees that $\Sigma(\mathfrak{g}, \mathfrak{a})$ is reduced.

**Corollary 3.4.** (cf. [B-Ø2]) *The Bessel function $F^\circ(\lambda, k, X)$ satisfies*

$$F^\circ(w\lambda, k, X) = F^\circ(\lambda, k, wX) = F^\circ(\lambda, k, X) \qquad \textit{for all } w \in \mathscr{W}_\Pi,$$
$$F^\circ(\lambda, k, 0) = 1.$$

The above corollary follows immediately from the fact that a similar statement for Heckman-Opdam hypergeometric functions holds, by taking the limit. As one can notice, we may derive at least the same amount of explicit information for the Bessel functions $F^\circ$ by a limit analysis of the hypergeometric functions $F$. Indeed, in [Ó-P], the authors were able to give an explicit formula for Heckman-Opdam hypergeometric functions when the root system $\mathscr{R}$ is reduced and $k \in \mathscr{Z}^+$, formula which we use to prove the theorem below.

For $X \in \mathfrak{a}$ write $\omega_k(X) := \prod_{\alpha\in\mathscr{R}^+} |\langle\alpha, X\rangle|^{2k_\alpha}$. Let $k \in \mathscr{Z}^+$, and set

$$\mathbb{D}(k) := \omega_k \mathbb{G}_+^\circ(k, 0).$$

***Example* 3.5.** If $k_\alpha = 1$ for all $\alpha$, then $\mathbb{D}(1) = (-1)^{|\mathscr{R}^+|} \prod_{\alpha\in\mathscr{R}^+} \breve{\alpha}\partial_\alpha$.

***Remark* 3.6.** The differential operator $\mathbb{D}(k)$ is obtained as the limit transition, under the weak topology, of $D(k) := \prod_{\alpha\in\mathscr{R}^+}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})^{2k_\alpha}\mathbb{G}_+(k, 0)$ after the deformation process. Further $\mathbb{D}(k) \in \mathbb{C}[\mathfrak{a}_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$. This follows from [Ó-P, Theorem 4.10], where the authors proved that $D(k) \in \mathbb{C}[A_\mathbb{C}] \otimes S(\mathfrak{a}_\mathbb{C})$.

**Theorem 3.7.** (cf. [B-Ø2]) *Assume that $k \in \mathscr{Z}^+$ and $\mathscr{R}$ is reduced. The Bessel functions $F^\circ$ are given by*

$$F^\circ(\lambda, k, X) = \frac{(-1)^{\sum_{\alpha>0} k_\alpha} 2^{-\sum_{\alpha>0} 2k_\alpha}}{\widetilde{c}(\rho(k), k)} \frac{\mathbb{D}(k)\Big(\sum_{w\in\mathscr{W}_\Pi} e^{w\lambda(X)}\Big)}{\prod_{\alpha\in\mathscr{R}^+}\langle\alpha, X\rangle^{2k_\alpha}\prod_{\alpha\in\mathscr{R}^+}\langle\alpha, \lambda\rangle^{2k_\alpha}},$$

*for all $(\lambda, X) \in \mathfrak{a}_\mathbb{C}^* \times \mathfrak{a}$.*

As we mentioned before, this class of Bessel functions encloses the spherical functions $\psi(\lambda, X)$ investigated in the previous section. In the light of Example 3.1, replacing $k_\alpha$ by $m_\alpha/2$ (and $\lambda$ by $i\lambda$) in Theorem 3.7 gives the explicit expressions of the spherical functions $\psi(\lambda, X)$. Below we give the list of all possible symmetric spaces $G/K$ where $m_\alpha \in 2\mathbb{N}$ (which in particular implies that $\Sigma(\mathfrak{g}, \mathfrak{a})$ is reduced), so that $k_\alpha \in \mathbb{N}$.

***Table I: Riemannian symmetric pairs with even multiplicity***

| $\mathfrak{g}$ | $\mathfrak{k}$ | $\Sigma$ | $m_\alpha$ | Comments |
|---|---|---|---|---|
| $\mathfrak{sl}(n,\mathbb{C})$ | $\mathfrak{su}(n)$ | $A_{n-1}$ | 2 | $n\geq 2$ |
| $\mathfrak{so}(2n+1,\mathbb{C})$ | $\mathfrak{so}(2n+1)$ | $B_n$ | 2 | $n\geq 2$ |
| $\mathfrak{sp}(n,\mathbb{C})$ | $\mathfrak{sp}(n)$ | $C_n$ | 2 | $n\geq 3$ |
| $\mathfrak{so}(2n,\mathbb{C})$ | $\mathfrak{so}(2n)$ | $D_n$ | 2 | $n\geq 4$ |
| $\mathfrak{so}(2n+1,1)$ | $\mathfrak{so}(2n+1)$ | $A_1$ | $2n$ | $n\geq 3$ |
| $\mathfrak{su}^*(2n)$ | $\mathfrak{sp}(n)$ | $A_{n-1}$ | 4 | $n\geq 2$ |
| $(\mathfrak{e}_6)_\mathbb{C}$ | $\mathfrak{e}_6$ | $E_6$ | 2 | – |
| $(\mathfrak{e}_7)_\mathbb{C}$ | $\mathfrak{e}_7$ | $E_7$ | 2 | – |
| $(\mathfrak{e}_8)_\mathbb{C}$ | $\mathfrak{e}_8$ | $E_8$ | 2 | – |
| $(\mathfrak{f}_4)_\mathbb{C}$ | $\mathfrak{f}_4$ | $F_4$ | 2 | – |
| $(\mathfrak{g}_2)_\mathbb{C}$ | $\mathfrak{g}_2$ | $G_2$ | 2 | – |
| $\mathfrak{e}_{6(-26)}$ | $\mathfrak{f}_{4(-20)}$ | $A_2$ | 8 | – |

***Table II: Special isomorphisms of Riemannian symmetric pairs with even multiplicity***

| $\mathfrak{g}$ | $\mathfrak{k}$ |
|---|---|
| $\mathfrak{sp}(1,\mathbb{C})\approx\mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{sp}(1)\approx\mathfrak{su}(2)$ |
| $\mathfrak{so}(3,\mathbb{C})\approx\mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{so}(3)\approx\mathfrak{su}(2)$ |
| $\mathfrak{sp}(2,\mathbb{C})\approx\mathfrak{so}(5,\mathbb{C})$ | $\mathfrak{sp}(2)\approx\mathfrak{so}(5)$ |
| $\mathfrak{so}(6,\mathbb{C})\approx\mathfrak{sl}(4,\mathbb{C})$ | $\mathfrak{so}(6)\approx\mathfrak{su}(4)$ |
| $\mathfrak{so}(3,1)\approx\mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{so}(3)\approx\mathfrak{su}(2)$ |
| $\mathfrak{so}(5,1)\approx\mathfrak{su}^*(4)$ | $\mathfrak{so}(5)\approx\mathfrak{sp}(2)$ |

In [O4, Theorem 3.15] and for $k\in\mathscr{K}^+$, Opdam proved that the hypergeometric functions can be written as $F(\lambda,k,a)=\frac{1}{|\mathscr{W}_\Pi|}\sum_{w\in\mathscr{W}_\Pi}G(w\lambda,k,a)$, where $G(\lambda,k,a)$ is an eigenfunction for the Dunkl-Cherednik operator $T(\xi,k)$ with eigenvalue $\lambda(\xi)$. Using Theorem 3.3, and the fact that $\lim_{\varepsilon\to 0}\varepsilon T^{(\varepsilon)}(\xi,k)=T^\circ(\xi,k)$, we obtain the following connection between the Bessel functions $F^\circ(\lambda,k,X)$ and the eigenfunctions of $T^\circ(\xi,k)$ with spectral parameter $\lambda$. The proof of the following theorem can be found in [B-Ø2], Theorem 4.6 and Proposition 4.7.

**Theorem 3.8.** (cf. [B-Ø2]) *Assume that $k\in\mathscr{Z}^+$.*

(i) *There exists a unique holomorphic function $G^\circ(\lambda,k,\,\cdot\,)$ in a tubular neighborhood $\mathfrak{u}$ of $\mathfrak{a}$ in $\mathfrak{a}_\mathbb{C}$ such that*

$$T^\circ(\xi,k)G^\circ(\lambda,k,X)=\lambda(\xi)G^\circ(\lambda,k,X),\qquad \xi\in\mathfrak{a}_\mathbb{C},$$
$$G^\circ(\lambda,k,0)=1.$$

(ii) *The Bessel functions can be written as*

$$F^\circ(\lambda,k,X)=\frac{1}{|\mathscr{W}_\Pi|}\sum_{w\in\mathscr{W}_\Pi}G^\circ(w\lambda,k,X).$$

(iii) *For all $w\in\mathscr{W}_\Pi$, $G^\circ(w\lambda,k,wX)=G^\circ(\lambda,k,X)$ and $G^\circ(\lambda,k,0)=1$.*

We should note that the above theorem (other than (ii)) is also proved in [O3], where the author uses a different approach. In [O3], the statement (ii) appears as the definition of the Bessel functions.

The eigenfunctions $G^\circ(\lambda, k, X)$ are known as the Dunkl kernels. These kernels play a major role in the theory of special functions related to Coxeter groups, which has had a rapid development in this area in the last few years. Among the broad literature in this direction, we refer to [Du], [Du-X], [He1], [J], [O3], [Ro], [B-Ø3], [B-Ø4], and [B-Ø5].

***Example* 3.9.** (Generalization of Example 2.5) Let us investigate the rank one case, and in order to be more convincing, we consider the non-reduced root system of type $BC_1$, i.e. $\mathscr{R} = \{\pm\alpha, \pm 2\alpha\}$. In this example we have $A_{\mathbb{C}} \simeq \mathbb{C}^*$ and $\mathbb{C}[A_{\mathbb{C}}] = \mathbb{C}[x^{-1}, x]$ where $x = e^\alpha$. The nontrivial Weyl group element acts by $x \mapsto x^{-1}$ on $A_{\mathbb{C}}$.
If $\xi = (2\alpha)\breve{}$, then $\partial_\xi = x\partial_x$. We will normalize the inner product on $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}^*_{\mathbb{C}}$ by $\langle\alpha, \alpha\rangle = 1$. In the $x$ coordinate, the differential operators (3.1) and (3.2) become

$$\Delta(k) = (x\partial_x)^2 - \left(k_\alpha \frac{1+x}{1-x} + 2k_{2\alpha}\frac{1+x^2}{1-x^2}\right) x\partial_x,$$

$$T(\xi, k) = x\partial_x + \left(\frac{k_\alpha}{1-x^{-1}} + \frac{2k_{2\alpha}}{1-x^{-2}}\right)(1-r) - (\frac{1}{2}k_\alpha + k_{2\alpha}),$$

where $r(x^m) = x^{-m}$. In this example, Opdam's shift operators are given in [He-S] by

$$\begin{array}{lll}
\mathbb{G}_+ & = \dfrac{x\partial_x}{x - x^{-1}}, & \text{with shift } (0,1); \\
\mathbb{G}_- & = (x^2-1)\partial_x + (k_\alpha + 2k_{2\alpha} - 1)(x + x^{-1}) + 2k_\alpha, & \text{with shift } (0,\text{-}1); \\
\mathbb{E}_+ & = \dfrac{1+x^{-1}}{1-x^{-1}} x\partial_x + \big(k_{2\alpha} - 1/2\big), & \text{with shift } (2,\text{-}1); \\
\mathbb{E}_- & = \dfrac{1-x^{-1}}{1+x^{-1}} x\partial_x + \big(k_\alpha + k_{2\alpha} - 1/2\big), & \text{with shift } (\text{-}2,1).
\end{array}$$

Let $z = -\frac{1}{4}x^{-1}(1-x)^2$ be a coordinate on $\mathscr{W}_\Pi \setminus A_{\mathbb{C}}$. Put $\gamma_1 = \lambda + \frac{1}{2}k_\alpha + k_{2\alpha}$, $\gamma_2 = -\lambda + \frac{1}{2}k_\alpha + k_{2\alpha}$, and $\gamma_3 = \frac{1}{2} + k_\alpha + k_{2\alpha}$. The functions $F(\lambda, k, a)$ and $G(\lambda, k, a)$ are given by

$$\begin{aligned}
F(\lambda, k, a) &= {}_2F_1(\gamma_1, \gamma_2; \gamma_3; z), \\
G(\lambda, k, a) &= {}_2F_1(\gamma_1, \gamma_2; \gamma_3; z) + \frac{\gamma_1}{4\gamma_3}(x - x^{-1})\,{}_2F_1(\gamma_1 + 1, \gamma_2 + 1; \gamma_3 + 1; z),
\end{aligned}$$

where ${}_2F_1(\gamma_1, \gamma_2; \gamma_3; z)$ is the Gauss hypergeometric function.
On $\mathfrak{a}_{\mathbb{C}}$, the infinitesimal operator (3.7) associated with $T(\xi, k)$ is given by

$$T^\circ(\xi, k) = \partial_X + \frac{k_\alpha + k_{2\alpha}}{X}(1 - r),$$

and the corresponding shift operators to $\mathbb{G}_\pm$ and $\mathbb{E}_\pm$ are the following

$$\begin{array}{lll}
\mathbb{G}^\circ_+ & = (2X)^{-1}\partial_X, & \text{with shift } (0,1); \\
\mathbb{G}^\circ_- & = (2X)\partial_X + 2(2k_\alpha + 2k_{2\alpha} - 1), & \text{with shift } (0,\text{-}1); \\
\mathbb{E}^\circ_+ & = 2X^{-1}\partial_X, & \text{with shift } (2,\text{-}1); \\
\mathbb{E}^\circ_- & = 2^{-1}X\partial_X + \big(k_\alpha + k_{2\alpha} - 1/2\big), & \text{with shift } (\text{-}2,1).
\end{array}$$

Using the fact that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left\{1 + \mathscr{O}(z^{-1})\right\}, \qquad \text{if } z \to \infty,$$

we obtain

$$\begin{aligned}F^\circ(\lambda,k,X) &= \lim_{\varepsilon\to 0} F\Big(\frac{\lambda}{\varepsilon},k,\exp(\varepsilon X)\Big)\\ &= \Gamma\Big(\frac{1}{2}+k_\alpha+k_{2\alpha}\Big)\Big(\frac{\lambda X}{2}\Big)^{\frac{1}{2}-k_\alpha-k_{2\alpha}} I_{k_\alpha+k_{2\alpha}-\frac{1}{2}}(\lambda X),\end{aligned}$$

where $I_\nu(z) = e^{-i\pi\nu/2}J_\nu(iz)$ and $J_\nu(z)$ is the Bessel function of the first kind. To regain Example 2.5 one needs to assume $k_\alpha = m_\alpha/2$ and $k_{2\alpha} = m_{2\alpha}/2$ (with the change of variable $\lambda \leftrightarrow i\lambda$). As we can see, the Bessel function $F^\circ$ generalizes $\psi(\lambda,t)$ from Example 2.5, since we do not constrain the root multiplicities $k_\alpha$ and $k_{2\alpha}$ to assume certain specific values. From classical analysis on special functions, it is a well-known fact that for fixed $\lambda\in\mathbb{C}$, the function $\zeta(X) := \Gamma(\frac{1}{2}+k_\alpha+k_{2\alpha})\left(\frac{\lambda X}{2}\right)^{\frac{1}{2}-k_\alpha-k_{2\alpha}} I_{k_\alpha+k_{2\alpha}-\frac{1}{2}}(\lambda X)$ is the unique analytic solution of the differential equation

$$\zeta'' + \frac{2(k_\alpha+k_{2\alpha})}{X}\zeta' = \lambda^2\zeta,$$

which is even and normalized by $\zeta(0)=1$. The eigenfunction $G^\circ(\lambda,k,X)$ for $T^\circ(\xi,k)$ is given by

$$\begin{aligned}G^\circ(\lambda,k,X) &= \lim_{\varepsilon\to \infty} G\Big(\frac{\lambda}{\varepsilon},k,\exp(\varepsilon X)\Big)\\ &= \Gamma\Big(\frac{1}{2}+k_\alpha+k_{2\alpha}\Big)\Big(\frac{\lambda X}{2}\Big)^{\frac{1}{2}-k_\alpha-k_{2\alpha}} \Big\{I_{k_\alpha+k_{2\alpha}-\frac{1}{2}}(\lambda X) + I_{k_\alpha+k_{2\alpha}+\frac{1}{2}}(\lambda X)\Big\}.\end{aligned}$$

## 4. Bessel functions associated with noncompact causal symmetric spaces

The theory of Harish-Chandra's spherical functions depends mainly on a compact subgroup $K$ of a Lie group $G$, i.e. a Cartan involution $\theta$; on the fact that the algebra $\mathscr{D}(G/K)$ contains an elliptic differential operator, and therefore all the joint eigenfunctions are real analytic; and finally on the Iwasawa decomposition $KAN$ and the Cartan decomposition $KAK$ of the Lie group $G$. Now, if we substitute $\theta$ by an arbitrary involution $\tau: G\to G$, and the subgroup $K$ by $H := \{h\in G \mid \tau(h)=h\}$, then $H$ is not compact in general, and there are no elliptic invariant differential operator on $G/H$. Further, $G\neq HAN$ and $G\neq HAH$. However, in [F-H-Ó] Faraut, Hilgert, and Ólafsson were able to prove that for noncompact causal symmetric spaces one can develop an analogue theory of spherical functions defined on open $H$-invariant domain of $G/H$. We refer to [H-Ó] for more details on the theory of causal symmetric spaces.

Let $(G,H)$ be a symmetric pair, that is $G$ is a connected semisimple Lie group with finite center, $H$ is a closed subgroup, and there exists an involutive automorphism $\tau$ of $G$ such that

$$(G^\tau)_0 \subset H \subset G^\tau,$$

where $G^\tau := \{\, g\in G \mid \tau(g)=g\,\}$, and $(G^\tau)_0$ is the identity component in $G^\tau$. Let $\mathfrak{g}$ be the Lie algebra of $G$, and let $\tau$ denote also the involution of $\mathfrak{g}$ obtained from that of $G$ by differentiation. Set $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ to be the decomposition of $\mathfrak{g}$ into $\pm1$ eigenspaces for $\tau$. Thus $\mathfrak{h}=\mathrm{Lie}(H)$, and $\mathfrak{q}$ can be identified with the tangent space of $G/H$ at $x_0=\mathbf{e}H$. Here $\mathbf{e}$ is the unit element in $G$.

Let $\theta$ be a Cartan involution such that $\theta\tau=\tau\theta$, and let $K$ be the corresponding maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}=\mathfrak{g}^\theta$. Put $\mathfrak{p}=\mathfrak{g}^{-\theta}$.

For $\mathfrak{u},\mathfrak{v}\subseteq\mathfrak{g}$, denote the centralizer of $\mathfrak{v}$ in $\mathfrak{u}$ by $\mathfrak{z}_\mathfrak{u}(\mathfrak{v})$. The symmetric pair $(\mathfrak{g},\tau)$ is called noncompactly causal if $\mathfrak{z}_{\mathfrak{p}\cap\mathfrak{q}}(\mathfrak{k}\cap\mathfrak{h})\neq\{0\}$. Thus there exists in $\mathfrak{p}\cap\mathfrak{q}$ a non-zero vector $X_0$ such that $\mathfrak{z}_{\mathfrak{p}\cap\mathfrak{q}}(\mathfrak{k}\cap\mathfrak{h})=\mathbb{R}X_0$. In particular, if $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$, then $X_0\in\mathfrak{a}$ and $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$.

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system and choose a positive system $\Sigma^+$ in $\Sigma$. We mention that for every noncompactly causal symmetric Lie algebra $(\mathfrak{g}, \tau)$, the root system $\Sigma$ is reduced. A root $\alpha \in \Sigma$ is called compact (resp. noncompact) if the root space $\mathfrak{g}^{(\alpha)} \subseteq \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q}$ (resp. $\mathfrak{g}^{(\alpha)} \subseteq \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h}$). Let $\Sigma_0$ and $\Sigma_n$ be the set of compact and noncompact roots, respectively. In particular $\Sigma = \Sigma_0 \cup \Sigma_n$.

Put as usual $\mathfrak{n} = \oplus_{\alpha \in \Sigma^+} \mathfrak{g}^{(\alpha)}$, and $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ where $m_\alpha = \dim(\mathfrak{g}^{(\alpha)})$. If $N := \exp(\mathfrak{n})$, then the map $H \times A \times N \ni (h, a, n) \mapsto han \in G$ is a diffeomorphism onto an open subset of $G$. Thus we may define the map $A : HAN \to A$ by $x \in HA(x)N$. Note that the map $A$ is left $H$-invariant.

Define the following open convex cone $c^0_{\max} = \{X \in \mathfrak{a} \mid \alpha(X) > 0 \ \forall \alpha \in \Sigma_n^+\}$, and write $S^0 = H \exp(c^0_{\max}) H$. In particular, if $C^0_{\max} := \mathrm{Ad}(H) c^0_{\max} \subset \mathfrak{q}$, then the mapping $H \times C^0_{\max} \to S^0, (h, X) \mapsto h \exp(X)$ is a homeomorphism.

For every $\alpha \in \Sigma$, consider $H_\alpha \in \{[X, \tau(X)] \mid X \in \mathfrak{g}^{(\alpha)}\} \subset \mathfrak{a}$ such that $\alpha(H_\alpha) = 2$. Denote by $\mathscr{E} = \{\lambda \in \mathfrak{a}_\mathbb{C} \mid \mathrm{Re}\lambda(H_\alpha) < 2 - m_\alpha \ \forall \alpha \in \Sigma_n^+\}$, where $m_\alpha = \dim(\mathfrak{g}^{(\alpha)})$. For $\lambda \in \mathscr{E}$, the spherical function $\varphi_\lambda$ is defined on $S^0$ by

$$\varphi_\lambda(x) = \int_H e^{\langle \lambda - \rho, A(xh) \rangle} \, dh, \tag{4.1}$$

(cf. [F-H-Ó]). (The measures are normalized via the Killing form.) We mention that $\varphi_\lambda$ is defined only on $S^0$.

For $\varepsilon > 0$, write $\gamma_\varepsilon = h \exp(\varepsilon X)$ with $h \in H$ and $X \in C^0_{\max}$. For arbitrary fixed $\lambda \in \mathscr{E}$, denote by

$$\psi(\lambda, X) := \lim_{\varepsilon \to 0} \varphi_{\frac{\lambda}{\varepsilon}}(\gamma_\varepsilon).$$

Using (4.1), the following integral representation of the Bessel functions associated with noncompact causal symmetric spaces holds.

**Theorem 4.1.** (cf. [B-Ø2]) *Let $G/H$ be a noncompact causal symmetric space, $\lambda \in \mathscr{E}$, and $X \in C^0_{\max}$. The limit $\psi(\lambda, X)$ and its derivatives exist. Its integral representation is given by*

$$\psi(\lambda, X) = \int_H e^{B(A_\lambda, \mathrm{Ad}(h) X)} \, dh.$$

Briefly we give the argument used to obtain the integral representation of $\psi(\lambda, X)$. Denote by $\mathbb{P} : \mathfrak{q} \to \mathfrak{a}$ the orthogonal projection on $\mathfrak{a}$. Notice that $A(\exp(\varepsilon X) h) = A(\exp(\varepsilon h^{-1} \cdot X))$, where $h^{-1} \cdot X := \mathrm{Ad}(h) X$. Write $h^{-1} \cdot X = \mathbb{P}(h^{-1} \cdot X) + Y \in \mathfrak{a} \oplus \mathfrak{a}^\perp$, where $\mathfrak{a}^\perp$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{q}$, and use the fact that $Y \in \mathfrak{a}^\perp$ can be written as $Y = Y_h + Y_n \in \mathfrak{h} \oplus \mathfrak{n}$. Then $h^{-1} \cdot X = \mathbb{P}(h^{-1} \cdot X) + Y_h + Y_n$. Now one may check that the functions $\varepsilon \mapsto A\big(\exp(\varepsilon Y_h) \exp(\varepsilon \mathbb{P}(h^{-1} \cdot X)) \exp(\varepsilon Y_n)\big)$ and $\varepsilon \mapsto A\big(\exp(\varepsilon h^{-1} \cdot X)\big)$ have the same derivative at $\varepsilon = 0$, and therefore

$$\lim_{\varepsilon \to 0} \frac{\lambda}{\varepsilon} A\big(\exp(\varepsilon h^{-1} \cdot X)\big) = \lambda \mathbb{P}\big(h^{-1} \cdot X\big).$$

This proof is similar to the argument in [D-R] that was done in the $G/K$ case.

***Example* 4.2.** Let $G = SO_0(1, n)$ and let $H = SO_0(1, n-1)$, $n \geq 2$. Let $\mathfrak{a} = \mathbb{R} X_0$ where $X_0 = E_{1,n+1} + E_{n+1,1}$. Here we use the standard notations for the matrix element $E_{i,j}$. In this case $\Sigma = \Sigma_n = \{\pm \alpha\}$. We choose the positive roots such that $\alpha(X_0) = 1$ and identify $\mathfrak{a}^*_\mathbb{C}$ with $\mathbb{C}$. Then $\rho = (n-1)/2$. For $t > 0$ and $\mathrm{Re}(\lambda) < -(n-3)/2$, we have

$$\begin{aligned} \varphi_\lambda(\exp(tX_0)) = \pi^{-1/2} 2^{n/2-1} e^{-i\pi(n/2-1)} \Gamma\Big(\frac{n}{2} - \frac{1}{2}\Big) (\operatorname{sh} t)^{-(n/2-1)} \\ \times \frac{\Gamma(-\lambda - \frac{n}{2} + \frac{3}{2})}{\Gamma(-\lambda + \frac{n}{2} - \frac{1}{2})} Q^{n/2-1}_{-\lambda-1/2}(\operatorname{ch} t), \end{aligned}$$

where $Q^\mu_\nu$ is the Legendre function of the second kind (cf. [F-H-Ó]). Using [Er, 3.2 (10)], we can rewrite the spherical function $\varphi_\lambda$ as $\varphi_\lambda(\exp(tX_0)) = \varphi^{(1)}_\lambda(\exp(tX_0)) + \varphi^{(2)}_\lambda(\exp(tX_0))$, where

$$\varphi^{(1)}_\lambda(\exp(tX_0)) = (i)^{\frac{n}{2}-1}\pi^{\frac{-1}{2}}2^{n-3}\Gamma\Big(\frac{n}{2}-\frac{1}{2}\Big)\Gamma\Big(\frac{n}{2}-1\Big)(\operatorname{sh} t)^{-(n-2)}\frac{\Gamma(-\lambda-\frac{n}{2}+\frac{3}{2})}{\Gamma(-\lambda+\frac{n}{2}-\frac{1}{2})}$$
$$\times {}_2F_1\Big(-\frac{\lambda}{2}-\frac{n}{4}+\frac{3}{4},\frac{\lambda}{2}-\frac{n}{4}+\frac{3}{4};-\frac{n}{2}+2;-\operatorname{sh}^2 t\Big),$$

and

$$\varphi^{(2)}_\lambda(\exp(tX_0)) = (i)^{\frac{n}{2}-1}\pi^{\frac{-1}{2}}2^{-1}\Gamma\Big(\frac{n}{2}-\frac{1}{2}\Big)\Gamma\Big(-\frac{n}{2}+1\Big)$$
$$\times {}_2F_1\Big(-\frac{\lambda}{2}+\frac{n}{4}-\frac{1}{4},\frac{\lambda}{2}+\frac{n}{4}-\frac{1}{4};\frac{n}{2};-\operatorname{sh}^2 t\Big).$$

Using the fact that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left\{1+\mathscr{O}(z^{-1})\right\}, \quad \text{if} \ \ z\to\infty,$$

we obtain

$$\lim_{\varepsilon\to 0}\varphi^{(1)}_{\frac{\lambda}{\varepsilon}}(\exp(\varepsilon tX_0)) = -(i)^{\frac{n}{2}-1}\pi^{\frac{1}{2}}2^{\frac{n}{2}-2}\Gamma\Big(\frac{n}{2}-\frac{1}{2}\Big)(\lambda t)^{-\frac{n}{2}+1}\frac{I_{1-\frac{n}{2}}(\lambda t)}{\sin(\pi(1-\frac{n}{2}))},$$

and

$$\lim_{\varepsilon\to 0}\varphi^{(2)}_{\frac{\lambda}{\varepsilon}}(\exp(\varepsilon tX_0)) = (i)^{\frac{n}{2}-1}\pi^{\frac{1}{2}}2^{\frac{n}{2}-2}\Gamma\Big(\frac{n}{2}-\frac{1}{2}\Big)(\lambda t)^{-\frac{n}{2}+1}\frac{I_{\frac{n}{2}-1}(\lambda t)}{\sin(\pi(1-\frac{n}{2}))}.$$

Here $I_\nu(z) = e^{-i\pi\nu/2}J_\nu(iz)$, where $J_\nu$ is the Bessel function of the first kind. In conclusion

$$\psi(\lambda,t) = \lim_{\varepsilon\to 0}\varphi_{\frac{\lambda}{\varepsilon}}(\exp(\varepsilon tX_0)) = (i)^{\frac{n}{2}-1}\pi^{-\frac{1}{2}}2^{\frac{n}{2}-1}\Gamma\Big(\frac{n}{2}-\frac{1}{2}\Big)(\lambda t)^{-\frac{n}{2}+1}\mathscr{K}_{1-\frac{n}{2}}(\lambda t),$$

where the function

$$\mathscr{K}_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\pi\nu)}$$

is the Bessel function of the third kind.

Recall that $\Sigma_0 = \Sigma(\mathfrak{k}\cap\mathfrak{h}\oplus\mathfrak{p}\cap\mathfrak{q},\mathfrak{a})$, and let $\Pi_0$ be the corresponding fundamental system of simple roots. Denote by $\mathscr{W}_{\Pi_0}$ the Weyl group generated by $\{r_\alpha \mid \alpha\in\Pi_0\}$, where $r_\alpha:\mathfrak{a}^*\to\mathfrak{a}^*$ is the reflection $r_\alpha(\lambda) = \lambda-\lambda(\breve{\alpha})\alpha$, and $\lambda(\breve{\alpha})$ was defined before.

In the next section we will investigate a more general class of Bessel functions related to arbitrary root subsystem, which will be called the $\Theta$-Bessel functions. This new class encloses both, the Bessel functions $F^\circ$ discussed in Section 3 and the present theory of Bessel functions associated with noncompact causal symmetric spaces. Indeed, in [B-Ø2] we were able to give explicit formulas for the $\Theta$-Bessel functions under certain conditions on the multiplicity functions. See the next section for more details. In particular, the general expression of the $\Theta$-Bessel functions for noncompact causal symmetric spaces reduces to:

**Theorem 4.3.** (cf. [B-Ø2]) *Let $G/H$ be a noncompact causal symmetric space such that $m_\alpha\in 2\mathbb{N}$ for all $\alpha\in\Sigma$. For $(\lambda,X)\in\mathscr{E}\times(\mathfrak{a}\cap C^0_{\max})$*

$$\int_H e^{B(A_\lambda,\operatorname{Ad}(h)X)}\,dh = c_0(m)\frac{\mathbb{D}(m/2)\Big(\sum_{w\in\mathscr{W}_{\Pi_0}}e^{w\lambda(X)}\Big)}{\prod_{\alpha\in\Sigma^+}\langle\alpha,X\rangle^{m_\alpha}\prod_{\alpha\in\Sigma^+}\langle\alpha,\lambda\rangle^{m_\alpha}},$$

*where $\mathbb{D}(m/2) = \prod_{\alpha\in\Sigma^+}|\langle\alpha,\cdot\rangle|^{m_\alpha}\mathbb{G}^\circ_+(m/2,0)$, and $c_0(m)$ is a constant that depends only on $m=(m_\alpha)_{\alpha\in\Sigma}$ and one may find it explicitly in Theorem 5.4 below (recall that for all noncompact causal symmetric spaces, $\Sigma$ is always reduced).*

See Table III and Table IV below for the list of all possible causal symmetric spaces for which the assumption of the above theorem holds.

Let $G$ be a connected semisimple Lie group such that $G_{\mathbb{C}}/G$ is ordered, i.e. $G/K$ is a bounded symmetric domain. This case is mainly characterized by the fact that $m_\alpha = 2$ for all $\alpha \in \Sigma$. Recalling the simple formula of the differential operator $\mathbb{D}(1)$ stated in Example 3.5, the above theorem gives a similar formula to the one for the character of discrete series representations of $G$.

**Theorem 4.4.** (cf. [B-Ø2]) *Let $G$ be a connected semisimple Lie group such that $G_{\mathbb{C}}/G$ is ordered. For $\lambda \in \mathscr{E}$ and $X \in C^0_{\max}$ there exists a positive constant $c_0$ such that*

$$\int_G e^{B(A_\lambda, \mathrm{Ad}(g)X)}\, dg = c_0 \frac{\sum_{w \in \mathscr{W}_{\Pi_0}} \varepsilon(w) e^{w\lambda(X)}}{\prod_{\alpha \in \Sigma^+} \langle \alpha, \lambda \rangle \prod_{\alpha \in \Sigma^+} \langle \alpha, X \rangle}.$$

We close this section by giving the list of all possible symmetric spaces $G/H$ where $m_\alpha \in 2\mathbb{N}$ for all $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$. This list has been extracted from the classification due to Oshima and Sekiguchi in [Os-Se] and to Hilgert and Ólafsson in [H-Ó].

***Table III: Noncompact causal symmetric pairs with even multiplicity***

| $\mathfrak{g}$ | $\mathfrak{h}$ | $\Sigma$ | $m_\alpha$ | Comments |
|---|---|---|---|---|
| $\mathfrak{sl}(n, \mathbb{C})$ | $\mathfrak{su}(n-j, j)$ | $A_{n-1}$ | 2 | $n \geq 2,\ 1 \leq j \leq [n/2]$ |
| $\mathfrak{so}(2n+1, \mathbb{C})$ | $\mathfrak{so}(2n-1, 2)$ | $B_n$ | 2 | $n \geq 2$ |
| $\mathfrak{sp}(n, \mathbb{C})$ | $\mathfrak{sp}(n, \mathbb{R})$ | $C_n$ | 2 | $n \geq 3$ |
| $\mathfrak{so}(2n, \mathbb{C})$ | $\mathfrak{so}(2n-2, 2)$ | $D_n$ | 2 | $n \geq 4$ |
| $\mathfrak{so}(2n, \mathbb{C})$ | $\mathfrak{so}^*(2n)$ | $D_n$ | 2 | $n \geq 5$ |
| $\mathfrak{so}(2n+1, 1)$ | $\mathfrak{so}(2n, 1)$ | $A_1$ | $2n$ | $n \geq 3$ |
| $\mathfrak{su}^*(2n)$ | $\mathfrak{sp}(n-j, j)$ | $A_{n-1}$ | 4 | $n \geq 2,\ 1 \leq j \leq [n/2]$ |
| $(\mathfrak{e}_6)_{\mathbb{C}}$ | $\mathfrak{e}_{6(-14)}$ | $E_6$ | 2 | – |
| $(\mathfrak{e}_7)_{\mathbb{C}}$ | $\mathfrak{e}_{7(-25)}$ | $E_7$ | 2 | – |
| $\mathfrak{e}_{6(-26)}$ | $\mathfrak{f}_{4(-20)}$ | $A_2$ | 8 | – |

***Table IV: Special isomorphisms of noncompact causal symmetric pairs with even multiplicity***

| $\mathfrak{g}$ | $\mathfrak{h}$ |
|---|---|
| $\mathfrak{sp}(1, \mathbb{C}) \approx \mathfrak{sl}(2, \mathbb{C})$ | $\mathfrak{sp}(1, \mathbb{R}) \approx \mathfrak{su}(1, 1)$ |
| $\mathfrak{so}(3, \mathbb{C}) \approx \mathfrak{sl}(2, \mathbb{C})$ | $\mathfrak{so}(1, 2) \approx \mathfrak{su}(1, 1)$ |
| $\mathfrak{sp}(2, \mathbb{C}) \approx \mathfrak{so}(5, \mathbb{C})$ | $\mathfrak{sp}(2, \mathbb{R}) \approx \mathfrak{so}(3, 2)$ |
| $\mathfrak{so}(6, \mathbb{C}) \approx \mathfrak{sl}(4, \mathbb{C})$ | $\mathfrak{so}(4, 2) \approx \mathfrak{su}(2, 2)$ |
| $\mathfrak{so}(6, \mathbb{C}) \approx \mathfrak{sl}(4, \mathbb{C})$ | $\mathfrak{so}^*(6) \approx \mathfrak{su}(3, 1)$ |
| $\mathfrak{so}(8, \mathbb{C}) = \mathfrak{so}(8, \mathbb{C})$ | $\mathfrak{so}^*(8) \approx \mathfrak{so}(2, 6)$ |
| $\mathfrak{so}(3, 1) \approx \mathfrak{sl}(2, \mathbb{C})$ | $\mathfrak{so}(2, 1) \approx \mathfrak{su}(1, 1)$ |
| $\mathfrak{so}(5, 1) \approx \mathfrak{su}^*(4)$ | $\mathfrak{so}(4, 1) \approx \mathfrak{sp}(1, 1)$ |

## 5. The Θ-Bessel functions

By introducing the so-called Θ-hypergeometric functions, in [P], Pasquale presents an extension of the theory of Heckman-Opdam hypergeometric functions associated with root systems, so that it encloses Harish-Chandra's theory of spherical functions and the theory of spherical functions on noncompact causal symmetric spaces. Using Pasquale's results, we introduce a new class of Bessel functions related to root systems, which we shall call the Θ-Bessel functions. The theory of Θ-Bessel functions extends naturally the theory of Bessel functions discussed in Section 3, and therefore it encloses the geometric case investigated in Section 2. It also covers the theory of Bessel functions associated with noncompact causal symmetric spaces.

The symbols $\mathscr{R}, \mathscr{R}^+, \mathfrak{a}, \mathfrak{a}_{\mathbb{C}}, A, A^+, \mathscr{W}_\Pi, \mathscr{K}, r_\alpha, \lambda(\breve{\alpha})$ will have the same meaning as in Section 3. Recall that $\mathscr{R}$ is supposed to be reduced and satisfies the crystallographic condition.

Let $\Pi = \{\alpha_1, \ldots, \alpha_N\}$ be the system of simple roots associated with $\mathscr{R}^+$. Let $\Theta \subset \Pi$ be an arbitrary subset of $\Pi$. The set $\langle\Theta\rangle$ of elements in $\mathscr{R}$, which can be written as linear combinations of elements from $\Theta$, is a subsystem of $\mathscr{R}$. Its Weyl group $\mathscr{W}_\Theta$ is generated by the reflections $r_{\alpha_j}$ with $\alpha_j \in \Theta$.

For a multiplicity function $k \in \mathscr{K}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we set

$$\begin{aligned} c_\Theta^+(\lambda, k) &= \prod_{\alpha \in \langle\Theta\rangle^+} \frac{\Gamma(\lambda(\breve{\alpha}))}{\Gamma(\lambda(\breve{\alpha}) + k_\alpha)}, \\ c_\Theta^-(\lambda, k) &= \prod_{\alpha \in \mathscr{R}^+ \setminus \langle\Theta\rangle^+} \frac{\Gamma(-\lambda(\breve{\alpha}) - k_\alpha + 1)}{\Gamma(-\lambda(\breve{\alpha}) + 1)}, \\ c_\Theta^{+,c}(\lambda, k) &= \prod_{\alpha \in \mathscr{R}^+ \setminus \langle\Theta\rangle^+} \frac{\Gamma(\lambda(\breve{\alpha}))}{\Gamma(\lambda(\breve{\alpha}) + k_\alpha)}, \end{aligned}$$

with the conventions

$$c_\emptyset^+ = c_\Pi^{+,c} = 1, \qquad \text{and} \qquad c_\Pi^- = 1.$$

If $\Theta = \Pi$, then $c_\Pi^+(\lambda, k)$ coincides with the $\widetilde{c}$-function (3.6).

Let $U$ be a connected and simply connected open subset of $\exp(i\mathfrak{a})$ containing the identity element. The function on $A^+U$ defined for generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by

$$F_\Theta(\lambda, k, a) := c_\Theta^-(\lambda, k) \sum_{w \in \mathscr{W}_\Theta} c_\Theta^+(w\lambda, k)\Phi(w\lambda, k, a), \quad a \in A^+U,$$

is called the Θ-hypergeometric function of spectral parameter $\lambda$ (see (3.5) for the definition of $\Phi(\lambda, k, a)$). We refer to [P] for more details on Θ-hypergeometric functions. As a linear combination of the Harish-Chandra series $\Phi(w\lambda, k, a)$, $F_\Theta(\lambda, k, a)$ is by construction a solution of the hypergeometric system (3.3).

***Example* 5.1.** When $\Theta = \Pi$, the ratio $F_\Pi(\lambda, k, a)/c_\Pi^+(\rho(k), k)$ coincides with the Heckman-Opdam hypergeometric function $F(\lambda, k, a)$. In the geometric case, the ratio coincides with Harish-Chandra's spherical function, up to the change of variable $\lambda \leftrightarrow i\lambda$.

***Example* 5.2.** We shall use the notation of the previous section. Let $\Pi_0$ be the fundamental system for the positive compact roots in $\Sigma_0^+$, and set $\Theta = \Pi_0$. Thus $\langle\Theta\rangle^+ = \Sigma_0^+$ and $W_\Theta = W_{\Pi_0}$. In this setting the ratio $F_{\Pi_0}(\lambda, k, a)/c_{\Pi_0}^+(\rho(k), k)c_{\Pi_0}^-(\rho(k), k)$ coincides with the spherical function $\varphi_\lambda(a)$ on $G/H$ investigated in the previous section.

Recall that $\mathscr{Z}^+$ denotes the set of positive integer-valued multiplicity functions, and that $\mathscr{Z}^+ \cap S = \emptyset$, where $S$ is the set of zeros of $\widetilde{c}(\rho(k), k) = c_\Pi^+(\rho(k), k)$.

Define

$$\mathfrak{a}_{\Theta,+} := \big\{H \in \mathfrak{a} \mid \alpha(H) > 0 \text{ for all } \alpha \in \mathscr{R}^+ \setminus \langle\Theta\rangle^+\big\}.$$

In [Ó-P, Theorem 5.1], Ólafsson and Pasquale give an explicit global formula for the $\Theta$-hypergeometric functions when $\mathscr{R}$ is reduced and $k \in \mathscr{K}^+$ by means of Opdam's shift operators $\mathbb{G}_+(\ell, k)$. We mention that, when $\mathscr{R}$ is reduced and $k \in \mathscr{K}^+$ (as we are assuming here), the $\lambda$-singularities of $F_\Theta$ are at most of first order and located along a locally finite family of hyperplanes in $\mathfrak{a}_\mathbb{C}$ (cf. [P, Theorem 3.5]). Using our previous results on the shift operators $\mathbb{G}_+^\circ(\ell, k)$ (see (3.10)) together with the explicit expressions of the $\Theta$-hypergeometric functions, we prove that for $X \in \mathfrak{a}_{\Theta,+}$ and $k \in \mathscr{K}^+$,

$$\widetilde{F}_\Theta^\circ(\lambda, k, X) := \lim_{\varepsilon \to 0} F_\Theta\Big(\frac{\lambda}{\varepsilon}, k, \exp(\varepsilon X)\Big)$$

exists. We will call $\widetilde{F}_\Theta^\circ$ the $\Theta$-Bessel functions, which are solutions for the Bessel system of differential equations (3.12). With the notations of Section 3, we have $F^\circ(\lambda, k, X) = \widetilde{F}_\Pi^\circ(\lambda, k, X)/c_\Pi^+(\rho(k), k)$ for $X \in \mathfrak{a}$ and $k \in \mathscr{K}^+$. The above transition relation linking $F^\circ$ to $\widetilde{F}_\Pi^\circ$ can be generalized by linking the $\Theta$-Bessel functions for arbitrary $\Theta$ to the Bessel functions $F^\circ$.

**Lemma 5.3.** (cf. [B-Ø2]) *For* $(\lambda, k, X) \in \mathfrak{a}_\mathbb{C}^* \times \mathscr{K}^+ \times \mathfrak{a}_{\Theta,+}$

$$F^\circ(\lambda, k, X) = \frac{(-1)^{d(\Theta,k)}}{c_\Pi^+(\rho(k), k)} \sum_{w \in \mathscr{W}_\Theta \backslash \mathscr{W}_\Pi} \widetilde{F}_\Theta^\circ(w\lambda, k, X),$$

*where* $d(\Theta, k) = \sum_{\alpha \in \mathscr{R}^+ \setminus \langle\Theta\rangle^+} k_\alpha$.

Recall the differential operator $\mathbb{D}(k) = \omega_k \mathbb{G}_+^\circ(k, 0)$ from Section 3, where $\omega_k(X) = \prod_{\alpha \in \mathscr{R}^+} |\langle \alpha, X\rangle|^{2k_\alpha}$. Indeed, we may rewrite $\mathbb{D}(k)$ as

$$\mathbb{D}(k) = \omega_k \mathbb{G}_+^\circ(k-1, 1) \circ \mathbb{G}_+^\circ(1, 0),$$

where

$$\mathbb{G}_+^\circ(1, 0) = (-1)^{|\mathscr{R}^+|}\omega_{-1/2} \prod_{\alpha \in \mathscr{R}^+} \frac{2\partial_\alpha}{\langle \alpha, \alpha\rangle}.$$

By means of [Ó-P, Theorem 5.1], we were able to prove in [B-Ø2, Theorem 4.9] the following explicit formulas.

**Theorem 5.4.** (cf. [B-Ø2]) *If* $k \in \mathscr{K}^+$ *and* $\mathscr{R}$ *is a reduced root system, then for* $(\lambda, X) \in \mathfrak{a}_\mathbb{C}^* \times \mathfrak{a}_{\Theta,+}$

$$\widetilde{F}_\Theta^\circ(\lambda, k, X) = (-1)^{d'(\Theta,k)} 2^{-\sum_{\alpha>0} 2k_\alpha} \frac{\mathbb{D}(k)\Big(\sum_{w \in \mathscr{W}_\Theta} e^{w\lambda(X)}\Big)}{\prod_{\alpha \in \mathscr{R}^+} \langle \alpha, X\rangle^{2k_\alpha} \prod_{\alpha \in \mathscr{R}^+} \langle \alpha, \lambda\rangle^{2k_\alpha}},$$

*where* $d'(\Theta, k) := \sum_{\alpha \in \langle\Theta\rangle^+} k_\alpha$. *We may also express* $\widetilde{F}_\Theta^\circ$ *in terms of an alternating series as*

$$\widetilde{F}_\Theta^\circ(\lambda, k, X) = (-1)^{d'(\Theta,k) + |\mathscr{R}^+|}\, 2^{\sum_{\alpha>0} 1 - 2k_\alpha} \frac{\widetilde{\mathbb{D}(k)}\Big(\sum_{w \in \mathscr{W}_\Theta} \varepsilon(w) e^{w\lambda(X)}\Big)}{\prod_{\alpha \in \mathscr{R}^+} \langle \alpha, X\rangle^{2k_\alpha} \prod_{\alpha \in \mathscr{R}^+} \langle \alpha, \lambda\rangle^{2k_\alpha - 1}},$$

*for all* $(\lambda, X) \in \mathfrak{a}_\mathbb{C}^* \times \mathfrak{a}_{\Theta,+}$, *and* $\widetilde{\mathbb{D}(k)} = \omega_k \mathbb{G}_+^\circ(k-1, 1) \circ \omega_{-1/2}$.

***Example* 5.5.** If $k \equiv 0$, then $\widetilde{F}_\Theta^\circ(\lambda, 0, X) = \sum_{w \in \mathscr{W}_\Theta} e^{w\lambda(X)}$.

***Example* 5.6.** If $k_\alpha = 1$ for all $\alpha \in \mathscr{R}^+$, then $d'(\Theta, k) = |\langle\Theta\rangle^+|$ and the differential operator $\widetilde{\mathbb{D}(1)} = \omega_{1/2}$. Thus when $\mathscr{R}$ is reduced

$$\widetilde{F}_\Theta^\circ(\lambda, 1, X) = (-1)^{|\mathscr{R}^+ \setminus \langle\Theta\rangle^+|} 2^{-|\mathscr{R}^+|} \frac{\sum_{w \in \mathscr{W}_\Theta} \varepsilon(w) e^{w\lambda(X)}}{\prod_{\alpha \in \mathscr{R}^+} \langle \alpha, X\rangle \prod_{\alpha \in \mathscr{R}^+} \langle \alpha, \lambda\rangle}.$$

(This generalizes Example 2.6 and Theorem 4.4, "the group case".)

## References


[Be-G-V] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, Berlin and New York, 1992

[Br-Hi] E. Brézin and S. Hikami, *Characteristic polynomials of real symmetric random matrices*, Comm. Math. Phys. **223** (2001), 363–382

[B-Ø1] S. Ben Saïd and B. Ørsted, *Analysis on flat symmetric spaces,* to appear in J. Math. Pures Appl. (2005)

[B-Ø2] ———, *Bessel functions for root systems via the trigonometric setting,* Int. Math. Res. Not. **9** (2005), p.551–585

[B-Ø3] ———, *Wave equations for Dunkl differential-difference operators,* to appear in Indag. Math. (2005)

[B-Ø4] ———, *Segal-Bargmann transforms associated with Coxeter groups,* to appear in Math. Ann. (2005)

[B-Ø5] ———, *On Fock spaces and $SL(2)$-triples for Dunkl operators,* to appear in Sémin. Congr., Soc. Math. France, Paris, (2005) (=the present proceedings)

[C] I. Cherednik, *A unification of Knizhnik-Zamolodchikov equations and Dunkl operators via affine Hecke algebras,* Invent. Math. **106** (1991), 411–432

[Cl] J.L. Clerc, *Une formule asymptotique du type Mehler-Heine pour les zonales d'un espace riemannien symétrique,* Studia Math. **57** (1976), 27–32.

[D-R] A. Dooley and J.W. Rice, *On contractions of semisimple Lie groups,* Trans. Amer. Math. Soc. **289** (1985), 185–202

[Du] C.F. Dunkl, *Differential-difference operators associated to reflection groups,* Trans. Amer. Math. Soc. **311** (1989), 167–183

[Du-X] C.F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables,* Cambridge Univ. Press, 2001

[Du-J-O] C.F. Dunkl, M. de Jeu, and E. Opdam, *Singular polynomials for finite reflection groups*, Trans. Amer. Math. Soc. **346** (1994), 237–256

[E-W] E. Inonu and E.P. Wigner, *On the contraction of groups and their representations*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 510–524

[Er] A. Erdelyi, *et. al.*, *Higher transcendental functions I,* Mc Graw-Hill, New York, 1953

[F-K] J. Faraut and A. Korányi, *Analysis on Symmetric Cones,* Clarendon Press, Oxford University Press, New York, 1994

[F-H-Ó] J. Faraut, J. Hilgert, and G. Ólafsson, *Spherical functions on ordered symmetric spaces,* Ann. Inst. Fourier (Grenoble) **44** (1994), 927–965

[HC1] Harish-Chandra, *Spherical functions on a semisimple Lie groups. I and II*, Amer. J. Math. **80** (1958), 241–310, 553–613

[HC2] ———, *A formula for semisimple Lie groups*, Amer. J. Math. **79** (1957), 733–760

[Hel] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces,* Academic Press, Inc., New York-London, 1978

[He1] G.J. Heckman, *A remark on the Dunkl differential-difference operators,* Harmonic analysis on reductive groups (Brunswick, ME, 1989), 181–191, Progr. Math. 101, Birkhäuser Boston, Boston, MA, 1991

[He2] ———, *Dunkl operators,* Séminaire Bourbaki, Vol. 1996/97, Astérisque **245** (1997), 223–246

[He-O] G.J. Heckman and E.M. Opdam, *Root systems and hypergeometric functions. I,* Compositio Math. **64** (1987), 329–352

[He-S] G.J. Heckman and H. Schlichtkrull, *Harmonic analysis and special functions on symmetric spaces,* Academic Press, Inc., San Diego, CA, 1994

[H-Ó] J. Hilgert and G. Ólafsson, *Causal symmetric spaces. Geometry and harmonic analysis,* Academic Press, Inc., San Diego, CA, 1997

[J] M.F. de Jeu, *The Dunkl transform,* Invent. Math. **113** (1993), 147–162

[Kn-Sa] F. Knop and S. Sahi, *Difference equations and symmetric polynomials defined by their zeros,* Int. Math. Res. Not. **10** (1996), 437–486

[M] I. G. Macdonald, *Symmetric functions and Hall polynomials,* The Clarendon Press, Oxford University Press, New York, 1995

[Ok] A. Okounkov, *(Shifted) Macdonald polynomials: q-integral representation and combinatorial formula,* Compositio Math. **112** (1998), 147–182

[Ok-Ol1] A. Okounkov and G. Olshanski, *Asymptotics of Jack polynomials as the number of variables goes to infinity,* Int. Math. Res. Not. **13** (1998), 641–682

[Ok-Ol2] ———, *Shifted Jack polynomials, binomial formula, and applications,* Math. Res. Lett. **4** (1997), 69–78

[Ol-Ve] G. Olshanski and A. Vershik, *Ergodic unitarily invariant measures on the space of infinite Hermitian matrices,* Contemporary Mathematical Physics (R. L. Dobrushin, R. A. Minlos, M. A. Shubin, A. M. Vershik, eds.), Amer. Math. Soc. Transl. Ser. 2 **175** (1996), 137–175

[Ols-Pe] M. A. Olshanetsky and A. M. Perelomov, *Completely integrable Hamiltonian systems connected with semisimple Lie algebras,* Invent. Math. **37** (1976), 93–108

[O1] E.M. Opdam, *Root systems and hypergeometric functions. III,* Compositio Math. **67** (1988), 21–49

[O2] ———, *Root systems and hypergeometric functions. IV,* Compositio Math. **67** (1988), 191–209

[O3] ———, *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group,* Compositio Math. **85** (1993), 333–373

[O4] ———, *Harmonic analysis for certain representations of graded Hecke algebras,* Acta Math. **175** (1995), 75–121

[Os-Se] T. Oshima and J. Sekiguchi, *Eigenspaces of invariant differential operators on an affine symmetric space,* Invent. Math. **57** (1980), 1–81

[Ó-P] G. Ólafsson and A. Pasquale, *A Paley-Wiener theorem for the* $\Theta$*-spherical transform: The even multiplicity case,* J. Math. Pures Appl. (9) **83** (2004), 869–927

[Ø-Z] B. Ørsted and G. Zhang, *Weyl quantization and tensor products of Fock and Bergman spaces,* Indiana Univ. Math. J. **43** (1994), 551–583

[P] A. Pasquale, *Asymptotic analysis of* $\Theta$*-hypergeometric functions*, Invent. Math. **157** (2004), 71–122

[Ro] M. Rösler, *Dunkl operators: theory and applications,* Orthogonal polynomials and special functions, E.K. Koelink and W. van Assche (eds.), Spring Lecture Notes in Math. **1817** (2003), 93–136

[S-V] A.N. Sergeev and A.P. Veselov, *Generalized discriminants, deformed quantum Calogero-Moser systems, and Jack polynomials,* arXiv:math-ph/0307036, version 2

[St] R. P. Stanley, *Some combinatorial properties of Jack functions,* Adv. Math. **77** (1989), 76–115

[Sta] R.J. Stanton, *Mean convergence of Fourier series on compact Lie groups*, Trans. Amer. Math. Soc. **218** (1976), 61–87

[Sw] P. Sawyer, *Spherical functions on symmetric cones,* Trans. Amer. Math. Soc. **349** (1997), 3569–3584

[T] C. Torossian, *Une application des opérateurs de Cherednik à l'isomorphisme d'Harish-Chandra pour les espaces symétriques,* C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), 139–144

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